Lie systems in Quantum Mechanics

José F. Cariñena
Universidad de Zaragoza
jfc@unizar.es

Nonlinear Integrable systems, Burgos, October 22-22, 2016
Abstract

After a quick presentation of the theory of Lie systems from a geometric perspective, recent progresses on their applications when compatible geometric structures exist will be described with a special emphasis in the particular case of admissible Kähler structures, and therefore with applications in Quantum Mechanics.
Outline

1. Introduction
2. Lie–Scheffers systems: a quick review
3. Some particular examples
4. The reduction method.
5. Structure preserving Lie systems
6. An example: Second order Riccati differential equation
7. Geometric approach to Quantum Mechanics
8. Lie Systems and Schrödinger equation
9. References
Solution of systems of differential equations appearing in many physical problems is not an easy task. In geometric terms they are represented by vector fields.

In order to find their solutions, i.e. the flow of the vector fields, as in a generic case there is not way of writing them in an explicit way, i.e. using fundamental functions, we are happy if, at least, we can express the solutions in terms of quadratures.

Characterisation of system admitting such type of solutions has received a lot of attention, and the answer is always based on the use of Lie algebras of vector fields containing the given one.

In general, in order to study such systems use is made of symmetry and reduction techniques. In such procedures the knowledge of some particular solutions of them or related systems may be useful.
For instance as far as Riccati equation is concerned one knows that

- If one particular solution is known, we can find the general solution by means of two quadratures.
- If two particular solutions are known, we can find the general solution by means of just one quadrature.
- If three particular solutions are known we can explicitly write the solution without any quadrature.

We are now interested in this particular kind of systems: the so called Lie systems. They appear very often in many problems in science and engineering.

We will fix our attention to the particular case of quantum mechanics, where they are useful in:

- Studying the time evolution of a quantum system
- In particular cases of time-independent Schrödinger equation
Lie–Scheffers systems: a quick review

Lie–Scheffers systems = Non-autonomous systems of first-order differential equations admitting a ...

Superposition rule: a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}$, $x = \Phi(u_1, \ldots, u_m; k_1, \ldots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

$$x(t) = \Phi(x(1)(t), \ldots, x(m)(t); k_1, \ldots, k_n),$$

with $\{x(a)(t) \mid a = 1, \ldots, m\}$ being a generic set of particular solutions of the system and where $k_1, \ldots, k_n$ are real numbers.

They are a generalisation of linear superposition rules for homogeneous linear systems for which $m = n$ and $x = \Phi(x(1), \ldots, x(n); k_1, \ldots, k_n) = k_1 x(1) + \cdots + k_n x(n)$ but

i) The number $m$ may be different from the dimension $n$.

ii) The function $\Phi$ is nonlinear in this more general case.
They appear quite often in many different branches of science ranging from pure mathematics to classical and quantum physics, control theory, economy, etc. Forgotten for a long time they had a revival due to the work of Winternitz and coworkers. One particular example is Riccati equation, of a fundamental importance in physics (for instance factorisation of second order differential operators, Darboux transformations and in general Supersymmetry in Quantum Mechanics) and in mathematics. These systems are related with equations in Lie groups and in general connections in fibre bundles.

In the solution of such non-autonomous systems of first-order differential equations we can use techniques imported from group theory, for instance Wei–Norman method, and reduction techniques coming from the theory of connections. Recent generalisations have also been shown to be useful for dealing with other systems of differential equations (e.g. Emden–Fowler equations, Abel equations). The existence of additional compatible geometric structures, like symplectic or Poisson structures may be useful in the search for solutions.
Lie–Scheffers theorem

**Theorem:** Given a non-autonomous system of $n$ first order differential equations

\[
\frac{dx^i}{dt} = X^i(x^1, \ldots, x^n, t), \quad i = 1 \ldots, n,
\]

a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$, $x = \Phi(u_1, \ldots, u_m; k_1, \ldots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

\[
x(t) = \Phi(x(1)(t), \ldots, x(m)(t); k_1, \ldots, k_n),
\]

with $\{x(a)(t) \mid a = 1, \ldots, m\}$ being a set of particular solutions of the system and where $k_1, \ldots, k_n$, are $n$ arbitrary constants, is that the system can be written as

\[
\frac{dx^i}{dt} = b_1(t)\xi_1^i(x) + \cdots + b_r(t)\xi_r^i(x), \quad i = 1 \ldots, n,
\]

where $b_1, \ldots, b_r$, are $r$ functions depending only on $t$ and $\xi_\alpha^i$, $\alpha = 1, \ldots, r$, are functions of $x = (x^1, \ldots, x^n)$, such that the $r$ vector fields in $\mathbb{R}^n$ given by

\[
X_\alpha \equiv \sum_{i=1}^{n} \xi_\alpha^i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i}, \quad \alpha = 1, \ldots, r,
\]
close on a real finite-dimensional Lie algebra, i.e. the $X_\alpha$ are l.i. and there are $r^3$ real numbers, $c_{\alpha\beta\gamma}$, such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X_\gamma.$$  

The number $r$ satisfies $r \leq mn$.

The geometric concept of superposition rule is the following:

A superposition rule for a $t$-dependent vector field $X$ in a $n$-dimensional manifold $M$ is a map $\Phi : M^{m} \times M \to M$ such that if $\{x(1)(t), \ldots, x(m)(t)\}$ is a generic set of integral curves of $X$, then $x(t) = \Phi(x(1)(t), \ldots, x(m)(t), k)$, with $k \in M$ is also integral curve of $X$, and each integral curve is obtained in this way.

The result of the Theorem in modern terms is that a $t$-dependent vector field $X$ admits a superposition rule if there exist $r$ fields $X_1, \ldots, X_r$ in $M$ and functions $b_1(t), \ldots, b_r(t)$ such that $X(x, t)$ be a linear combination

$$X(x, t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x).$$  

The $t$-dependent vector field can be seen as a family of vector fields $\{X_t \mid t \in \mathbb{R}\}$. 
Definition. The minimal Lie algebra of a given a \( t \)-dependent vector field \( X \) on a manifold \( M \) is the smallest real Lie algebra, \( V^X \), containing the vector fields \( \{X_t\}_{t \in \mathbb{R}} \), namely \( V^X = \text{Lie}(\{X_t \mid t \in \mathbb{R}\}) \).

Definition. The vector field associated to a non-autonomous system \( X \) allows us to define a generalised distribution \( \mathcal{D}^X : x \in M \mapsto \mathcal{D}^X_x \subset TM \), where \( \mathcal{D}_x = \{Y_x \mid Y \in V^X\} \subset T_x M \), and \( X \) also gives rise to a generalised co-distribution \( \mathcal{V} : x \in M \mapsto \mathcal{V}_x \subset T^* M \), where \( \mathcal{V}_x = \{\omega_x \mid \omega_x(Y_x) = 0, \forall Y_x \in \mathcal{D}^X_x \} \).

Remark that the Lie–Scheffers theorem can be reformulated as follows:

**Theorem:** A system \( X \) admits a superposition rule if and only if the minimal Lie algebra \( V^X \) is finite-dimensional.

Definition. A function \( f : U \subset U^X \to \mathbb{R} \) is a local first integral (or \( t \)-independent constant of the motion) for a given \( t \)-dependent vector field \( X \) over \( \mathbb{R}^n \) if \( Xf = 0 \)

Then \( f \) is a first integral if and only if \( df \in V^X \mid_U \).

One can easily prove that:
Property. Given a $t$-dependent vector field $X$ on a $n$-dimensional manifold $M$ and a point $x \in U^X$ where the rank of $D^X$ is equal to $k$, the associated co-distribution $\mathcal{V}^X$ admits, in a neighbourhood of $x$, a local basis of the form, $df_1, \ldots, df_{n-k}$, where, $f_1, \ldots, f_{n-k}$, is a family of first integrals of $X$. Additionally, the space $\mathcal{I}^X|_U$ of first-integrals of the system $X$ over an open $U$ of $M$, can be put in the form

$$\mathcal{I}^X|_U = \{ g \in C^\infty(U) \mid \exists F : U \subset \mathbb{R}^{n-k} \to \mathbb{R}, \ g = F(f_1, \ldots, f_{n-k}) \}.$$

There exist different procedures to derive superposition rules for Lie systems. We can use a method based on the diagonal prolongation notion.

Definition. Given a $t$-dependent vector field $X$ over $M$, its diagonal prolongation to $M^{m+1}$ is the $t$-dependent vector field $\tilde{X}$ over $M^{m+1}$ such that

- $\tilde{X}$ projects onto $X$ by the map $\text{pr} : (x_0, \ldots, x_m) \in M^{m+1} \mapsto x_0 \in M$, that is, $\text{pr}_* \tilde{X} = X$.
- $\tilde{X}$ is invariant under permutation $x_i \leftrightarrow x_j$, with $i, j = 0, \ldots, m$. 

The procedure to determine superposition rules described is:

i) Take a basis $X_1, \ldots, X_r$ of the Vessiot–Guldberg Lie algebra $V$ associated with the Lie system.

ii) Choose the minimum integer $m$ such that the diagonal prolongations to $M^m$ of the elements of the previous basis are linearly independent at a generic point.

ii) Obtain $n$ common first-integrals for the diagonal prolongations, $\tilde{X}_1, \ldots, \tilde{X}_r$, to $M^{m+1}$ (for instance, by means of the method of characteristics).

iii) Obtain the expression of the variables of one of the spaces $M$ only in terms of the other variables of $M^{m+1}$ and the above mentioned $n$ first-integrals.

The so obtained expressions give rise to a superposition rule in terms of any generic family of $m$ particular solutions and $n$ constants corresponding to the possible values of the derived first-integrals.
Some particular examples

A) Inhomogeneous linear systems:

\[
\frac{dx^i}{dt} = \sum_{j=1}^{n} A^{i j}(t) x^j + B^i(t), \quad i = 1, \ldots, n.
\]

It is related with the \((n^2 + n)\)-dimensional Lie algebra of the affine group.

In this case \(r = n^2 + n\) and \(m = n + 1\) and the equality \(r = mn\) also follows. The superposition function \(\Phi : \mathbb{R}^{n(n+1)} \to \mathbb{R}^n\) is:

\[
x = \Phi(u_1, \ldots, u_{n+1}; k_1, \ldots, k_n) = u_1 + k_1(u_2 - u_1) + \cdots + k_n(u_{n+1} - u_1).
\]

B) The Riccati equation \((n = 1)\)

\[
\frac{dx(t)}{dt} = a_2(t) x^2(t) + a_1(t) x(t) + a_0(t).
\]

Now \(m = r = 3\) and the superposition principle comes from the relation

\[
\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k,
\]
or in other words,

\[
x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) + k x_2(t)(x_1(t) - x_3(t))}{(x_3(t) - x_2(t)) + k (x_1(t) - x_3(t))}.
\]

The value \( k = \infty \) must be accepted, otherwise we do not obtain the solution \( x_2 \).

The associated Lie algebra is \( \mathfrak{sl}(2, \mathbb{R}) \).

C) Lie–Scheffers systems on Lie groups

Consider a basis of are either left–invariant (or right–invariant) vector fields \( X_{\alpha} \) in \( G \) as corresponding to the Lie algebra \( \mathfrak{g} \) of \( G \) or its opposite algebra.

If \( \{a_1, \ldots, a_r\} \) is a basis for the tangent space \( T_eG \) and \( X^R_{\alpha} \) denotes the right-invariant vector field in \( G \) such that \( X^R_{\alpha}(e) = a_{\alpha} \), a Lie–Scheffers system is

\[
\dot{g}(t) = - \sum_{\alpha=1}^{r} b_{\alpha}(t) X^R_{\alpha}(g(t)).
\]

When applying \( (R_{g(t)}^{-1})^*g(t) \) to both sides we obtain the equation on \( T_eG \)

\[
(R_{g(t)}^{-1})^*g(t)(\dot{g}(t)) = - \sum_{\alpha=1}^{r} b_{\alpha}(t)a_{\alpha}, \quad (**)
\]
This is usually written with a slight abuse of notation:

\[(\dot{g} g^{-1})(t) = -\sum_{\alpha=1}^{r} b\alpha(t)a\alpha.\]

Such equation is right-invariant. Then,

If \(\bar{g}(t)\) is a solution of \((**\) with initial condition \(\bar{g}(0) = e\), the solution \(g(t)\) with initial conditions \(g(0) = g_0\) is given by \(\bar{g}(t)g_0\).

Moreover, there is a superposition rule \(\Phi : G \times G \rightarrow G\) involving one solution

\[\Phi(g, g_0) = g g_0.\]

This example is very useful because there are many other examples related with them as explained next.
D) Lie-Scheffers systems on homogeneous spaces for Lie groups

Let $H$ be a closed subgroup of $G$ and consider the homogeneous space $M = G/H$. The right–invariant vector fields $X_R^\alpha$ are $\tau$-projectable and the $\tau$-related vector fields in $M$ are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of $G$ on $M$.

$$\tau^* g X_R^\alpha (g) = -X_\alpha (gH) ,$$

and we will have an associated Lie–Scheffers system on $M$:

$$X(x, t) = \sum_{\alpha=1}^r b_{\alpha}(t) X_\alpha(x) .$$

Therefore, a solution of this last system starting from $x_0$ will be:

$$x(t) = \Phi(g(t), x_0) ,$$

with $g(t)$ being a solution of (**).

The converse property is true: Given a Lie Scheffers system defined by complete vector fields with associated Lie algebra $\mathfrak{g}$, we can see these as fundamental vector fields relative to an action which can be found by integrating the vector fields.
The reduction method

The important ingredient is an equation on a Lie group

\[ \dot{g}(t) g(t)^{-1} = a(t) = - \sum_{\alpha=1}^{r} b_{\alpha}(t) a_{\alpha} \in T_e G , \]  

(●)

with \( g(0) = e \in G \), which may be solved by a method extending the so-called Wei–Norman method.

It may happen that the only different from zero coefficients are those corresponding to a subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \). Then the equation reduces to a simpler equation on a subgroup, involving less coordinates.

The fundamental result is that if we know a particular solution of the problem associated in a homogeneous space, the original solution reduces to one on the subgroup.

Let us choose a curve \( g'(t) \) in the group \( G \), and define the curve \( \bar{g}(t) \) by \( \bar{g}(t) = g'(t)g(t) \). The new curve in \( G \), \( \bar{g}(t) \), determines a new Lie system.
Indeed,

\[ R_{\bar{g}(t)}^{-1} * \bar{g}(t)(\dot{\bar{g}}(t)) = R_{g'(t)}^{-1}(t) * g'(t)(\dot{g}'(t)) - \sum_{\alpha=1}^{r} b_{\alpha}(t) \text{Ad}(g'(t))a_{\alpha}, \]

which is an equation similar to the original one but with a different right hand side.

In this way we can define an action of the group of curves in the Lie group \( G \) on the set of Lie systems on the group. This can be used to reduce a given Lie system to a simpler one.

The aim is to choose the curve \( g'(t) \) in such a way that the new equation be simpler. For instance, we can choose a subgroup \( H \) and look for a choice of \( g'(t) \) such that the right hand side lies in \( T_{e}H \), and hence \( \bar{g}(t) \in H \) for all \( t \).

If \( \Psi : G \times M \to M \) is a transitive action of \( G \) on a homogeneous space \( M \), which can be identified with the set \( G/H \) of left-cosets, by choosing a fixed point \( x_0 \), then the integral curves starting from the point \( x_0 \) associated to both Lie systems are related by

\[ \bar{x}(t) = \Psi(\bar{g}(t), x_0) = \Psi(g'(t)g(t), x_0) = \Psi(g'(t), x(t)). \]
Therefore, this gives an action of the group of curves in $G$ on the set of associated Lie systems in homogeneous space $s$.

More explicitly, if we consider a curve $g'(t)$ in the group, the Lie system transforms into a new one

$$\dot{x} = \sum_{\alpha=1}^{r} \bar{b}_\alpha(t)X_\alpha(x),$$

in which

$$\bar{b} = \text{Ad} (g'(t))b(t) + \dot{g}' g'^{-1}.$$

The important result is that the knowledge of a particular solution of the associated Lie system in $G/H$ allows us to reduce the problem to one in the subgroup $H$.

**Theorem:** Each solution of (●) on the group $G$ can be written in the form $g(t) = g_1(t) h(t)$, where $g_1(t)$ is a curve on $G$ projecting onto a solution $\tilde{g}_1(t)$ for the left action $\lambda$ on the homogeneous space $G/H$ and $h(t)$ is a solution of an equation but for the subgroup $H$, given explicitly by

$$(\dot{h} h^{-1})(t) = -\text{Ad} (g_1^{-1}(t)) \left( \sum_{\alpha=1}^{r} b_\alpha(t)a_\alpha + (\dot{g}_1 g_1^{-1})(t) \right) \in T_e H.$$
There are particularly interesting cases in which the manifold $M$ is endowed with additional structures. For instance, let $(M, \Omega)$ be a symplectic manifold and the vector fields arising in the expression of the $t$-dependent vector field describing a Lie system are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra.

These vector fields correspond to a symplectic action of the Lie group $G$ on $(M, \Omega)$. The Hamiltonian functions of such vector fields, defined by $i(X_\alpha)\Omega = -dh_\alpha$, do not close on the same Lie algebra under Poisson bracket, but we can only say that

$$d \left( \{h_\alpha, h_\beta\} - h_{[X_\alpha, X_\beta]} \right) = 0,$$

and then they span a Lie algebra extension of the original one.

The important fact is that we can define a $t$-dependent Hamiltonian

$$h_t = \sum_{\alpha} b_\alpha(t) h_\alpha,$$
with the functions $h_\alpha$ closing a Lie algebra, in such a way that $i(X_t)\Omega = -dh_t$.

As an example we can consider the differential equation of an $n$-dimensional Winternitz–Smorodinsky oscillator of the form

$$
\begin{cases}
\dot{x}_i = p_i, \\
\dot{p}_i = -\omega^2(t)x_i + \frac{k}{x_i^3},
\end{cases}
\quad i = 1, \ldots, n.
$$

which describes the integral curves of the $t$-dependent vector field on $T^*\mathbb{R}^n$

$$
X_t = \sum_{i=1}^{n} \left[ p_i \frac{\partial}{\partial x_i} + \left( -\omega^2(t)x_i + \frac{k}{x_i^3} \right) \frac{\partial}{\partial p_i} \right],
$$

which can be written as $X_t = X_2 + \omega^2(t)X_1$ with $X_1, X_2$ and $X_3 = -[X_1, X_2]$ being given by

$$
X_1 = -\sum_{i=1}^{n} x_i \frac{\partial}{\partial p_i}, \quad X_2 = \sum_{i=1}^{n} \left( p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3} \frac{\partial}{\partial p_i} \right), \quad X_3 = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i} \right).
$$

Note that $X_t$ is a Lie system, because $X_1, X_2$ and $X_3$ close on a $\mathfrak{sl}(2, \mathbb{R})$ algebra:

$$
[X_1, X_2] = -X_3, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2.
$$
Moreover, the preceding vector fields are Hamiltonian vector fields with respect to the usual symplectic form \( \omega_0 = \sum_{i=1}^{n} dx^i \wedge dp_i \) with Hamiltonian functions

\[
\begin{align*}
    h_1 &= \frac{1}{2} \sum_{i=1}^{n} x_i^2, \\
    h_2 &= \frac{1}{2} \sum_{i=1}^{n} \left( p_i^2 + \frac{k}{x_i^2} \right), \\
    h_3 &= \sum_{i=1}^{n} x_ip_i,
\end{align*}
\]

which obey that

\[
\begin{align*}
    \{ h_1, h_2 \} &= h_3, \\
    \{ h_1, h_3 \} &= -h_1, \\
    \{ h_2, h_3 \} &= h_2.
\end{align*}
\]

Consequently, every curve \( h_t \) taking values in the Lie algebra \((W, \{\cdot, \cdot\})\) spanned by \( h_1, h_2 \) and \( h_3 \) gives rise to a Lie system which is Hamiltonian in \( T^*\mathbb{R}^n \) with respect to the symplectic structure \( \omega_0 \) in such a way that

\[
X_t = X_2 + \omega^2(t)X_1 = \widehat{\omega}_0^{-1}(dh_2 + \omega^2(t)dh_1),
\]

i.e. the Hamiltonian is \( h_t = h_2 + \omega^2(t)h_1 \).

We can go a step further and consider Lie systems in (may be degenerate) Poisson manifolds, or even more generally in Dirac manifolds.
An example: Second order Riccati differential equation

The usual Riccati equation comes from reduction of a linear differential equation by taking into account the invariance under dilations of such equations.

Starting from

\[ A_3 \dddot{y} + A_2 \ddot{y} + A_1 \dot{y} + A_0 y = 0 \]

where we can assume that \( A_3(t) > 0 \), and writing \( y = e^u \), with \( x = \dot{u} \) we arrive to

\[ A_3(\dddot{x} + 3 \dot{x} \ddot{x} + x^3) + A_2(\ddot{x} + x^2) + A_1 x + A_0 = 0. \]

This is the second order Riccati equation.

It has been shown (JFC+ MF Rañada+M Santander, JMP 46, 062703 (2005)) that such second-order Riccati equations admits a Lagrangian of the form:

\[ L(t, x, v) = \frac{1}{v + U(t, x)}, \]

with \( U(t, x) = a_0(t) + a_1(t)x + a_2(t)x^2. \)
The corresponding \( t \)-dependent Hamiltonian obtained from the Legendre transformation

\[
p = \frac{\partial L}{\partial v} = -\frac{1}{(v + U(t,x))^2} \implies v = \frac{1}{\sqrt{-p}} - U(t,x),
\]

i.e. the image is the open submanifold \( \mathcal{O} = \{ (x,p) \in T^*_x\mathbb{R} \mid p < 0 \} \) and we can define in \( \mathcal{O} \) the Hamiltonian

\[
h(t,x,p) = p \left( \frac{1}{\sqrt{-p}} - U(t,x) \right) - \sqrt{-p} = -2\sqrt{-p} - pU(t,x).
\]

Consequently, the Hamilton equations for \( h \) are

\[
\begin{align*}
\dot{x} &= \frac{\partial h}{\partial p} = \frac{1}{\sqrt{-p}} - U(t,x), \\
\dot{p} &= -\frac{\partial h}{\partial x} = p \frac{\partial U}{\partial x}(t,x).
\end{align*}
\]

which, taking into account the form of \( U(t,x) \) turn out to be

\[
\begin{align*}
\dot{x} &= \frac{\partial h}{\partial p} = \frac{1}{\sqrt{-p}} - a_0(t) - a_1(t)x - a_2(t)x^2, \\
\dot{p} &= -\frac{\partial h}{\partial x} = p(a_1(t) + 2a_2(t)x).
\end{align*}
\]
This is a Lie system: In fact, consider the set of vector fields

\[ X_1 = \frac{1}{\sqrt{-p}} \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p}, \]
\[ X_4 = x^2 \frac{\partial}{\partial x} - 2xp \frac{\partial}{\partial p}, \quad X_5 = \frac{x}{\sqrt{-p}} \frac{\partial}{\partial x} + 2\sqrt{-p} \frac{\partial}{\partial p}. \]

The time-dependent vector field describing the system is

\[ X(t, x) = X_1 - a_0(t)X_2 - a_1(t)X_3 - a_2(t)X_4, \]

and the vector fields close on the commutation relations

\[
[X_1, X_2] = 0, \quad [X_1, X_3] = \frac{1}{2}X_1, \quad [X_1, X_4] = X_5, \quad [X_1, X_5] = 0, \\
[X_2, X_3] = X_2, \quad [X_2, X_4] = 2X_3, \quad [X_2, X_5] = X_1, \\
[X_3, X_4] = X_4, \quad [X_3, X_5] = \frac{1}{2}X_5, \\
[X_4, X_5] = 0.
\]

and then we see that it is a Lie system related to a Vessiot-Guldberg Lie algebra of vector fields \( V \).
More specifically, the vector fields $X_1, \ldots, X_5$ span a five dimensional Lie algebra of vector fields $V$ that is not solvable because $[V, V] = V$.

Moreover, $V$ is not a semisimple algebra. It admits an Abelian solvable ideal $V_1 = \langle X_1, X_5 \rangle$, and $V_2 = \langle X_2, X_3, X_4 \rangle$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Therefore $V$ is a semidirect sum $V_1 \oplus_s V_2$.

Consequently, the Lie algebra $V$ gives rise to a Lie group of the form $G = \mathbb{R}^2 \ltimes SL(2, \mathbb{R})$, where $\ltimes$ denotes the semidirect product of $SL(2, \mathbb{R})$ by $\mathbb{R}^2$, and a Lie group action $\Phi : G \times \mathcal{O} \to \mathcal{O}$ whose fundamental vector fields are those of $V$.

This action enables us to put the general solution $\xi(t)$ of the system of Hamilton equations for the second order Riccati equation in the form $\xi(t) = \Phi(g(t), \xi_0)$, where $g(t)$ is the solution of the equation

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{5} b_{\alpha}(t) X^{R}_{\alpha}(g), \quad g(0) = e,$$

on $G$, with the $X^{R}_{\alpha}$ being a family of right-invariant vector fields over $G$ such that the $X^{R}_{\alpha}(e) \in T_e G$ close the same commutation relations as the $X_{\alpha}$. 
To be remarked that the vector fields $X_i$ here considered are Hamiltonian with respect to the usual symplectic form in $T^*\mathbb{R}$, their hamiltonians being respectively given by:

$$
    h_1 = 2\sqrt{-p}, \quad h_2 = -p, \quad h_3 = -xp, \quad h_4 = -x^2p,
$$

and it turns out that their nonvanishing Poisson brackets are

$$
\{h_1, h_3\} = \frac{1}{2}h_1, \quad \{h_1, h_4\} = h_5, \quad \{h_1, h_5\} = 2, \quad \{h_2, h_3\} = h_2,
$$

$$
\{h_2, h_4\} = 2h_3, \quad \{h_2, h_5\} = h_1, \quad \{h_3, h_4\} = h_4, \quad \{h_3, h_5\} = \frac{1}{2}h_5
$$

with $h_5 = 2x\sqrt{-p}$. They close on a six-dimensional real Lie algebra with the function $h_6 = 1$. Moreover, it can be seen that the $t$-dependent system can be put into the form $\hat{\Lambda}(-dh_t)$, where $h_t$ is a $t$-parametrized family of functions over $O$ of the form $h_t = h_1 - a_0(t)h_2 - a_1(t)h_3 - a_2h_4$ and therefore the Lie system we are considering is Hamiltonian.

This shows that we can find a superposition rule for the second order Riccati equation which can be obtained using the general theory through the common first-integrals for the appropriated diagonal prolongations $\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5$ on a certain $O^{(m)} \subset T^*\mathbb{R}^{(m)}$. 
The Schrödinger picture of Quantum mechanics admits a geometric interpretation similar to that of classical mechanics.

A separable complex Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) can be considered as a real linear space, to be then denoted \(\mathcal{H}_R\). The norm in \(\mathcal{H}\) defines a norm in \(\mathcal{H}_R\), where \(\|\psi\|_R = \|\psi\|_C\).

The linear real space \(\mathcal{H}_R\) is endowed with a natural symplectic structure as follows:

\[
\omega(\psi_1, \psi_2) = 2 \text{Im} \langle \psi_1, \psi_2 \rangle.
\]

The Hilbert \(\mathcal{H}_R\) can be considered as a real manifold modelled by a Banach space admitting a global chart.

The tangent space \(T_{\phi} \mathcal{H}_R\) at any point \(\phi \in \mathcal{H}_R\) can be identified with \(\mathcal{H}_R\) itself: the isomorphism associates \(\psi \in \mathcal{H}_R\) with the vector \(\dot{\psi} \in T_{\phi} \mathcal{H}_R\) given by:

\[
\dot{\psi} f(\phi) := \left. \left( \frac{d}{dt} f(\phi + t\psi) \right) \right|_{t=0}, \quad \forall f \in C^\infty(\mathcal{H}_R).
\]
The real manifold can be endowed with a symplectic 2-form $\omega$:

$$\omega(\dot{\psi}, \dot{\psi}') = 2 \text{Imag} \langle \psi, \psi' \rangle.$$ 

One can see that the constant symplectic structure $\omega$ in $\mathcal{H}_\mathbb{R}$, considered as a Banach manifold, is exact, i.e., there exists a 1-form $\theta \in \bigwedge^1(\mathcal{H}_\mathbb{R})$ such that $\omega = -d\theta$. Such a 1-form $\theta \in \bigwedge^1(\mathcal{H})$ is, for instance, the one defined by

$$\theta(\psi_1)[\dot{\psi}_2] = -\text{Imag} \langle \psi_1, \psi_2 \rangle.$$ 

This shows that the geometric framework for usual Schrödinger picture is that of symplectic mechanics, as in the classical case.

A continuous vector field in $\mathcal{H}_\mathbb{R}$ is a continuous map $X: \mathcal{H}_\mathbb{R} \rightarrow \mathcal{H}_\mathbb{R}$. For instance for each $\phi \in \mathcal{H}$, the constant vector field $X_\phi$ defined by

$$X_\phi(\psi) = \dot{\phi}.$$ 

It is the generator of the one-parameter subgroup of transformations of $\mathcal{H}_\mathbb{R}$ given by

$$\Phi(t, \psi) = \psi + t \phi.$$
As another particular example of vector field consider the vector field $X_A$ defined by the $\mathbb{C}$-linear map $A : \mathcal{H} \to \mathcal{H}$, and in particular when $A$ is skew-selfadjoint.

With the natural identification natural of $T\mathcal{H}_\mathbb{R} \cong \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}$, $X_A$ is given by

$$X_A : \phi \mapsto (\phi, A\phi) \in \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}.$$ 

When $A = I$ the vector field $X_I$ is the Liouville generator of dilations along the fibres, $\Delta = X_I$, usually denoted $\Delta$ given by $\Delta(\phi) = (\phi, \phi)$.

Given a selfadjoint operator $A$ in $\mathcal{H}$ we can define a real function in $\mathcal{H}_\mathbb{R}$ by

$$a(\phi) = \langle \phi, A\phi \rangle,$$

i.e.,

$$a = \langle \Delta, X_A \rangle.$$

Then,

$$da_\phi(\psi) = \frac{d}{dt} a(\phi + t\psi)_{t=0} = \frac{d}{dt} [\langle \phi + t\psi, A(\phi + t\psi) \rangle]_{t=0}
= 2 \text{Re} \langle \psi, A\phi \rangle = 2 \text{Imag} \langle -i A\phi, \psi \rangle = \omega(-i A\phi, \psi).$$

If we recall that the Hamiltonian vector field defined by the function $a$ is such that for each $\psi \in T_\phi \mathcal{H} = \mathcal{H},$

$$da_\phi(\psi) = \omega(X_a(\phi), \psi),$$
we see that
\[ X_a(\phi) = -i A\phi. \]

Therefore if \( A \) is the Hamiltonian \( H \) of a quantum system, the Schrödinger equation describing time-evolution plays the rôle of ‘Hamilton equations’ for the Hamiltonian dynamical system \((\mathcal{H}, \omega, h)\), where \( h(\phi) = \langle \phi, H\phi \rangle \): the integral curves of \( X_h \) satisfy
\[ \dot{\phi} = X_h(\phi) = -i H\phi. \]

The real functions \( a(\phi) = \langle \phi, A\phi \rangle \) and \( b(\phi) = \langle \phi, B\phi \rangle \) corresponding to two selfadjoint operators \( A \) and \( B \) satisfy
\[ \{a, b\}(\phi) = -i \langle \phi, [A, B]\phi \rangle, \]

because
\[ \{a, b\}(\phi) = [\omega(X_a, X_b)](\phi) = \omega_\phi(X_a(\phi), X_b(\phi)) = 2 \text{Imag} \langle A\phi, B\phi \rangle, \]

and taking into account that
\[ 2 \text{Imag} \langle A\phi, B\phi \rangle = -i [\langle A\phi, B\phi \rangle - \langle B\phi, A\phi \rangle] = -i [\langle \phi, AB\phi \rangle - \langle \phi, B\phi \rangle], \]

we find the above result.
In particular, on the integral curves of the vector field $X_h$ defined by a Hamiltonian $H$,

$$\dot{a}(\phi) = \{a, h\}(\phi) = -i \langle \phi, [A, H]\phi \rangle,$$

what is usually known as Ehrenfest theorem:

$$\frac{d}{dt} \langle \phi, A\phi \rangle = -i \langle \phi, [A, H]\phi \rangle.$$

There is another relevant symmetric $(0, 2)$ tensor field which is given by the Real part of the inner product. It endows $\mathcal{H}_{\mathbb{R}}$ with a Riemann structure and we have also a complex structure $J$ such that

$$g(v_1, v_2) = -\omega(Jv_1, v_2), \quad \omega(v_1, v_2) = g(Jv_1, v_2),$$

together with

$$g(Jv_1, Jv_2) = g(v_1, v_2), \quad \omega(Jv_1, Jv_2) = \omega(v_1, v_2).$$

The triplet $(g, J, \omega)$ defines a Kähler structure in $\mathcal{H}_{\mathbb{R}}$ and the symmetry group of the theory must be the unitary group $U(\mathcal{H})$ whose elements preserve the inner product, or in an alternative but equivalent way (in the finite-dimensional case), by the intersection of the orthogonal group $O(2n, \mathbb{R})$ and the symplectic group $Sp(2n, \mathbb{R})$. 
The time evolution from time $t_0$ to time $t$, even in the non-autonomous case, is described in terms of the evolution operator $U(t, t_0)$:

$$\psi(t) = U(t, t_0)\psi(t_0)$$

It must be a symmetry of the theory, i.e. for each fixed $t_0$, $U(t, t_0)$ is a curve in the unitary group $U(\mathcal{H})$.

Assume by simplicity that $\mathcal{H}$ is finite-dimensional, and then as

$$\frac{dU(t, t_0)}{dt} \in T_{U(t, t_0)}U(\mathcal{H}) \implies \frac{dU(t, t_0)}{dt}(U(t, t_0))^{-1} \in T_IU(\mathcal{H}) \approx u(\mathcal{H}),$$

and therefore, there exists a curve $H(t)$ in $\text{Herm}(n, \mathbb{C})$ such that

$$\frac{dU(t, t_0)}{dt} = -iH(t)U(t, t_0).$$

In this equation $H(t)$ does not depend on $t_0$ because of the relation

$$U(t, t_0) = U(t, t_1)U(t_1, t_0),$$

which implies

$$\frac{dU(t, t_0)}{dt}(U(t, t_0))^{-1} = \frac{dU(t, t_1)}{dt}(U(t, t_1))^{-1}$$
This is a Lie system in the unitary group $U(\mathcal{H})$ with associated Lie algebra $u(\mathcal{H})$ in the most general case. Sometimes however we can deal with some of its subalgebras.

Every curve $H(t)$ in $u(\mathcal{H})$ can be written as a linear combination of at most $n^2$ elements, those of a basis of $u(\mathcal{H})$, and therefore these (finite-dimensional) quantum systems are Lie systems.

As the elements of the Vessiot-Guldberg Lie algebra are skew-Hermitians, all of them define simultaneously Hamiltonian vector fields and Killing vector fields, and the system is a Lie-Kähler system.

As an example consider a Hamiltonian operator $H(t)$ that can be written as a linear combination, with some $t$-dependent real coefficients $b_1(t), \ldots, b_r(t)$, of some Hermitian operators,

$$H(t) = \sum_{k=1}^{r} b_k(t) H_k ,$$

where the $H_k$ form a basis of a real finite-dimensional Lie algebra $V$ relative to the Lie bracket of observables, i.e. $[H_j, H_k] = \sum_{l=1}^{r} i c_{jkl} H_l$, with $c_{jkl} \in \mathbb{R}$ and $j, k, l = 1, \ldots, r$. 
It determines a \( t \)-dependent Schrödinger equation

\[
\frac{d\psi}{dt} = -iH(t)\psi = -i \sum_{k=1}^{r} b_k(t)H_k\psi.
\]

The vector fields \( X_k \) such that \( X_k(\psi) = -i H_k \psi \) are such that the \( t \)-dependent vector field \( X \) corresponding to the equation is \( X = \sum_{k=1}^{r} b_k(t)X_k \) and

\[
[X_j, X_k] = -\sum_{l=1}^{r} c_{jkl}X_l, \quad j, k = 1, \ldots, r.
\]

As an instance, if \( \mathcal{H} = \mathbb{C}^2 \), the time evolution is described by a curve \( -iH(t) := \dot{U}_t U_t^{-1} \) in the Lie algebra \( \mathfrak{u}(2) \) of \( U(2) \). Using the basis

\[
I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

and denoting \( \mathbf{S} = (\sigma_1, \sigma_2, \sigma_3)/2 \) and \( \mathbf{B} := (B_1, B_2, B_3) \), the Hamiltonian can be written as

\[
H(t) := B_0(t)I_0 + \mathbf{B}(t) \cdot \mathbf{S}.
\]
Using the identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, the Schrödinger equation is

\[
\begin{pmatrix}
\dot{q}_1 \\
\dot{p}_1 \\
\dot{q}_2 \\
\dot{p}_2
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
0 & 2B_0(t) + B_3(t) & -B_2(t) & B_1(t) \\
-2B_0(t) - B_3(t) & 0 & -B_1(t) & -B_2(t) \\
B_2(t) & B_1(t) & 0 & 2B_0(t) - B_3(t) \\
-B_1(t) & B_2(t) & B_3(t) - 2B_0(t) & 0
\end{pmatrix} \begin{pmatrix}
q_1 \\
p_1 \\
q_2 \\
p_2
\end{pmatrix},
\]

while the vector fields are now

\[
X_0 = -\Gamma = p_1 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial p_2},
\]
\[
X_1 = \frac{1}{2} \left( p_2 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_2} - q_1 \frac{\partial}{\partial p_2} \right),
\]
\[
X_2 = \frac{1}{2} \left( -q_2 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial p_1} + q_1 \frac{\partial}{\partial q_2} + p_1 \frac{\partial}{\partial p_2} \right),
\]
\[
X_3 = \frac{1}{2} \left( p_1 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial p_2} \right)
\]

satisfying

\[
[X_0, \cdot] = 0, \quad [X_1, X_2] = -X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2.
\]
The vector fields $X_0, X_1, X_2, X_3$ are Hamiltonian with Hamiltonian functions given by

$$h_0(\psi) = \frac{1}{2} \langle \psi, \psi \rangle = \frac{1}{2} (q_1^2 + p_1^2 + q_2^2 + p_2^2),$$

$$h_1(\psi) = \frac{1}{2} \langle \psi, S_1 \psi \rangle = \frac{1}{2} (q_1 q_2 + p_1 p_2),$$

$$h_2(\psi) = \frac{1}{2} \langle \psi, S_2 \psi \rangle = \frac{1}{2} (q_1 p_2 - p_1 q_2),$$

$$h_3(\psi) = \frac{1}{2} \langle \psi, S_3 \psi \rangle = \frac{1}{4} (q_1^2 + p_1^2 - q_2^2 - p_2^2).$$

$h_1, h_2, h_3$ are functionally independent, but $h_0^2 = 4(h_1^2 + h_2^2 + h_3^2)$.

When $\mathcal{H}$ is not finite-dimensional Lie system theory applies when the $t$-dependent Hamiltonian can be written as a linear combination with $t$-dependent coefficients of Hamiltonians $H_i$ closing on, under the commutator bracket, a real finite-dimensional Lie algebra.

Note however that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension.
On the other hand, as the fundamental concept for measurements is the expectation value of observables, two vector fields such that

\[
\frac{\langle \psi_2, A\psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle} = \frac{\langle \psi_1, A\psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle}, \quad \forall A \in \text{Her}(\mathcal{H})
\]

should be considered as indistinguishable.

This is only possible when \( \psi_2 \) is proportional to \( \psi_1 \), and therefore we must consider rays rather than vectors the elements describing the quantum states.

The space of states is not \( \mathbb{C}^n \) but the projective space \( \mathbb{CP}^{n-1} \).

It is possible to define a Kähler structure on \( \mathbb{CP}^{n-1} \) and therefore to study Lie-Kähler systems leading to superposition rules and to study time evolution in this projective space.
A linear SODE in normal form $\phi'' = b_1(x)\phi + b_2(x)\phi'$ can be written in the form of a system of two first-order differential equations in the variables $(v_\phi, \phi)$:

$$\begin{cases} v'_\phi = b_2(x)v_\phi + b_1(x)\phi \\ \phi' = v_\phi \end{cases}$$

Identifying $\mathbb{R}^2$ with $T\mathbb{R}$, $(v_\phi, \phi)$ are bundle coordinates, the preceding system determines the integral curves of the $x$-dependent vector field

$$X = v_\phi \frac{\partial}{\partial \phi} + (b_1(x)\phi + b_2(x)v_\phi) \frac{\partial}{\partial v_\phi},$$

which is said to be a SODE vector field because of the coefficient of $\partial/\partial \phi$.

The linear system determining its integral curves is

$$\begin{pmatrix} v'_\phi \\ \phi' \end{pmatrix} = \begin{pmatrix} b_2(x) & b_1(x) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_\phi \\ \phi \end{pmatrix}.$$  

The projection onto $\mathbb{R}$ of such curves are solutions of the differential equation

$$\phi'' = b_2(x)\phi' + b_1(x)\phi.$$
We are mainly interested in equations of Schrödinger type, those with $b_2(x) \equiv 0$.

The corresponding vector field is a linear combination $X = b_1(x)X_1 - X_3$ where

$$X_1 = \phi \frac{\partial}{\partial v\phi}, \quad X_3 = -v\phi \frac{\partial}{\partial \phi},$$

which together with

$$X_2 = \frac{1}{2} \left( v\phi \frac{\partial}{\partial v\phi} - \phi \frac{\partial}{\partial \phi} \right),$$

close on a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$:

$$[X_1, X_3] = 2X_2, \quad [X_1, X_2] = X_1, \quad [X_3, X_2] = -X_3.$$  \hfill (4)

Therefore Schrödinger type equations and the corresponding linear systems are Lie systems with Vessiot-Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

Vector fields $X_1, X_2$ and $X_3$ are fundamental vector fields corresponding to the linear action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$.

The map $F : \mathbb{R}^2_* \rightarrow \mathbb{R}$ defined by $F(x, y) = x/y$ is equivariant with respect to the restriction of the linear action $\Phi$ of $SL(2, \mathbb{R})$ on $\mathbb{R}^2_*$ and the action $\Psi$ of $SL(2, \mathbb{R})$ on
\( \mathbb{R} \), or even better on the real projective line \( \mathbb{RP}^1 = \overline{\mathbb{R}} = \mathbb{R} \cup \{ \infty \} \) by linear fractional transformations, i.e. \( \Psi : SL(2, \mathbb{R}) \times \mathbb{RP}^1 \to \mathbb{RP}^1 \) is defined by

\[
\Psi(A, u) = \frac{\alpha}{\gamma} u + \frac{\beta}{\gamma} u + \delta, \quad \text{if} \quad u \neq -\frac{\delta}{\gamma},
\]

\[
\Psi(A, \infty) = \frac{\alpha}{\gamma}, \quad \Psi\left(A, -\frac{\delta}{\gamma}\right) = \infty,
\]

\[
A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R}).
\]

Equivariance means that \( F \circ \Phi_A = \Psi_A \circ F \). The corresponding fundamental vector fields of the action \( \Psi \) are now

\[
\overline{X}_1 = \frac{\partial}{\partial u}, \quad \overline{X}_2 = u \frac{\partial}{\partial u}, \quad \overline{X}_3 = u^2 \frac{\partial}{\partial u},
\]

and as \( F \) is equivariant, the fundamental vector fields associated to \( \Phi \) and \( \Psi \) are \( F \)-related, i.e. \( \overline{X}_i = F_*(X_i) \), \( i = 1, 2, 3 \), and then a system defined by the vector fields \( \overline{X}_i \) is a Lie system corresponding to a Riccati equation.

The image under \( F \) of an integral curve of the \( x \)-dependent vector field \( X = b_1(x) X_1 + b_2(x) X_2 + b_3(x) X_3 \), which is a linear system, is an integral curve of \( \tilde{X} = \tilde{b}_1(x) X_1 + \tilde{b}_2(x) X_2 + \tilde{b}_3(x) X_3 \), i.e. a solution of the corresponding Riccati equation.


THANKS FOR YOUR ATTENTION !!!

TANTI AUGURI A TE!!!