On a solution of Zamolodchikov’s tetrahedron equation

Adam Doliwa
doliwa@matman.uwm.edu.pl

University of Warmia and Mazury, Olsztyn, Poland

Workshop Nonlinear Integrable Systems
Burgos, 20 – 22 October 2016
OUTLINE

1 Desargues maps and tetrahedron equation - affine version
   - Desargues maps and non-commutative KP systems
   - The normalization map and the Veblen map
   - The ten-term relation

2 Non-commutative tetrahedron map and its quantum reduction
   - Normalization map and its ultralocal/quantum reduction
   - The Veblen map and its quantum reduction
   - Quantum map with Zamolodchikov’s tetrahedron property
1. **Desargues Maps and Tetrahedron Equation - Affine Version**
   - Desargues maps and non-commutative KP systems
   - The normalization map and the Veblen map
   - The ten-term relation

2. **Non-commutative Tetrahedron Map and Its Quantum Reduction**
   - Normalization map and its ultralocal/quantum reduction
   - The Veblen map and its quantum reduction
   - Quantum map with Zamolodchikov’s tetrahedron property
Maps $\phi : \mathbb{Z}^N \rightarrow \mathbb{P}^M(\mathbb{D})$, such that the points $\phi(n), \phi(i)(n)$ and $\phi(j)(n)$ are collinear, for all $n \in \mathbb{Z}^N$, $i \neq j$; here $\mathbb{D}$ is a division ring.

**Notation:** $\phi(i)(n_1, \ldots, n_i, \ldots, n_N) = \phi(n_1, \ldots, n_i + 1, \ldots, n_N)$

In homogeneous coordinates $\phi : \mathbb{Z}^N \rightarrow \mathbb{D}_*^{M+1}$

$$\phi + \phi(i)A_{ij} + \phi(j)A_{ji} = 0, \quad i \neq j,$$

where $A_{ij} : \mathbb{Z}^K \rightarrow \mathbb{D}_*$.

The compatibility condition of the above linear system reads

$$A_{ij}^{-1}A_{ik} + A_{kj}^{-1}A_{ki} = 1,$$

$$A_{ik(j)}A_{jk} = A_{jk(i)}A_{ik}, \quad i, j, k \text{ distinct}$$
In non-homogeneous (affine) coordinates we have $\Phi: \mathbb{Z}^N \rightarrow \mathbb{D}^M$

$$(\Phi(j) - \Phi) = (\Phi(i) - \Phi)B_{ij},$$

- the first part of the compatibility condition gives
  $$B_{ij}B_{jk} = B_{ik},$$
  which allows to introduce a potential $\sigma: \mathbb{Z}^N \rightarrow \mathbb{D}_*$ such that
  $$B_{ij} = \sigma(i)\sigma_{(j)}^{-1};$$

- the second part of the compatibility condition takes then the form
  $$\sigma_{(i)}^{−1} - \sigma_{(j)}^{−1})\sigma_{(ij)} + (\sigma_{(j)}^{−1} - \sigma_{(k)}^{−1})\sigma_{(jk)} + (\sigma_{(k)}^{−1} - \sigma_{(i)}^{−1})\sigma_{(ki)} = 0$$

known as the non-commutative discrete mKP system [*Nijhoff, Capel 1990*]
**The linear problem**

\[
\begin{align*}
\Phi(2) - \Phi(1) &= (\Phi - \Phi(1))x_1, \\
\Phi(23) - \Phi(12) &= (\Phi(2) - \Phi(12))x_2, \\
\Phi(3) - \Phi(2) &= (\Phi - \Phi(2))x_3
\end{align*}
\]

\[
\begin{align*}
\Phi(23) - \Phi(13) &= (\Phi(3) - \Phi(13))\tilde{x}_1, \\
\Phi(3) - \Phi(1) &= (\Phi - \Phi(1))\tilde{x}_2, \\
\Phi(13) - \Phi(12) &= (\Phi(1) - \Phi(12))\tilde{x}_3
\end{align*}
\]

**Proposition**  
[AD, Kashaev]

The birational map \( R: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X} \times \mathcal{X} \)

\[
\begin{align*}
\tilde{x}_1 &= [x_3 + x_1(1 - x_3)]^{-1} x_1 x_2, \\
\tilde{x}_2 &= x_3 + x_1(1 - x_3), \\
\tilde{x}_3 &= 1 + (x_2 - 1) [(1 - x_1)x_3 + x_1(1 - x_2)]^{-1} (x_3 + x_1(1 - x_3)),
\end{align*}
\]

satisfies Zamolodchikov’s tetrahedron condition

\[
R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123},
\]
4D CUBE VISUALIZATION OF ZAMOLODCHIKOV’S CONDITION

\[ R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123} \]
4D consistency of Desargues maps is a consequence of the Desargues theorem
Decomposition into pentagonal maps

The map \( R : \mathcal{X}^3 \to \mathcal{X}^3 \) can be decomposed as

\[
R = P_{23} \circ V_{12} \circ N_{13}
\]

where

- the (affine) normalization map \( N : (x_1, x_2) \to (x'_1, x'_2) \)

\[
x'_1 = (x_2 + x_1 - x_1 x_2)^{-1} x_1, \quad x'_2 = x_2 + x_1 - x_1 x_2,
\]

\( N \) satisfies the pentagonal condition

\[
N_{12} \circ N_{13} \circ N_{23} = N_{23} \circ N_{12}
\]

- the Veblen map \( V : (x_1, x_2) \to (\bar{x}_1, \bar{x}_2) \)

\[
\bar{x}_1 = x_1 x_2, \quad \bar{x}_2 = (1 - x_1) x_2 (1 - x_1 x_2)^{-1},
\]

\( V \) satisfies the reversed pentagonal condition

\[
V_{23} \circ V_{13} \circ V_{12} = V_{12} \circ V_{23}
\]

- transposition \( P : (x_1, x_2) \to (x_2, x_1) \)
Given four collinear points $A$, $B$, $C$ and $D$, consider two pairs of linear relations between their non-homogeneous coordinates

\[
\begin{align*}
\phi_A - \phi_B &= (\phi_C - \phi_B)x_1 \\
\phi_D - \phi_A &= (\phi_C - \phi_A)x_2
\end{align*}
\]

and

\[
\begin{align*}
\phi_A - \phi_B &= (\phi_D - \phi_B)x'_1 \\
\phi_D - \phi_B &= (\phi_C - \phi_B)x'_2.
\end{align*}
\]

The (affine) normalization map is a consequence of that change

\[
\begin{align*}
x'_1 &= (x_2 + x_1 - x_1 x_2)^{-1} x_1, \\
x'_2 &= x_2 + x_1 - x_1 x_2,
\end{align*}
\]
The Veblen \((6_2, 4_3)\) configuration

\[
\begin{align*}
\phi_{AC} - \phi_{AD} &= (\phi_{AB} - \phi_{AD})x_1 \\
\phi_{BC} - \phi_{CD} &= (\phi_{AC} - \phi_{CD})x_2
\end{align*}
\]

and

\[
\begin{align*}
\phi_{BC} - \phi_{BD} &= (\phi_{AB} - \phi_{BD})\bar{x}_1 \\
\phi_{BD} - \phi_{CD} &= (\phi_{AD} - \phi_{CD})\bar{x}_2
\end{align*}
\]

\[
\bar{x}_1 = x_1 x_2, \quad \bar{x}_2 = (1 - x_1)x_2(1 - x_1 x_2)^{-1},
\]
THEOREM

[Kashaev, Sergeev 1998]

Given a solution $N$ of the functional pentagon equation, and given a solution $V$ of the reversed functional pentagon equation on the same set $\mathcal{X}$, then the map $R = P_{23} \circ V_{12} \circ N_{13}$ satisfies the Zamolodchikov tetrahedron equation, provided

$$V_{13} \circ N_{12} \circ V_{14} \circ N_{34} \circ V_{24} = N_{34} \circ V_{24} \circ N_{14} \circ V_{13} \circ N_{12}.$$  

Start from seven points (black circles) of the star configuration $(10_2, 5_4)$ AND FOUR CORRESPONDING LINEAR RELATIONS there are two distinct ways to complete the configuration using the normalization and Veblen flips.
1. DESARGUES MAPS AND TETRAHEDRON EQUATION - AFFINE VERSION
   - Desargues maps and non-commutative KP systems
   - The normalization map and the Veblen map
   - The ten-term relation

2. NON-COMMUTATIVE TETRAHEDRON MAP AND ITS QUANTUM REDUCTION
   - Normalization map and its ultralocal/quantum reduction
   - The Veblen map and its quantum reduction
   - Quantum map with Zamolodchikov’s tetrahedron property
The normalization map in homogeneous coordinates

\[ \begin{align*}
\phi_A &= \phi_C x_1 + \phi_B y_1, \\
\phi_D &= \phi_C x_2 + \phi_A y_2,
\end{align*} \]

Proposition [AD, Sergeev 2014]

The normalization map \( N : [(x_1, y_1), (x_2, y_2)] \rightarrow [(x'_1, y'_1), (x'_2, y'_2)] \)

\[ \begin{align*}
x'_1 &= (x_2 + x_1 y_2)^{-1} x_1, \\
x'_2 &= x_2 + x_1 y_2,
\end{align*} \]

\[ \begin{align*}
y'_1 &= y_1 x_1^{-1} x_2 (x_2 + x_1 y_2)^{-1} x_1, \\
y'_2 &= y_1 y_2
\end{align*} \]

satisfies the pentagon relation

\[ N_{12} \circ N_{13} \circ N_{23} = N_{23} \circ N_{12} \]
Quantum reduction of the normalization map

Observation
In the commutative case the normalization map provides a Poisson automorphism of the field of rational functions $\mathbb{k}(x_1, y_1, x_2, y_2)$ equipped with the Poisson structure

$$\{x_i, y_i\} = x_i y_i, \quad \{x_i, x_j\} = \{y_i, y_j\} = \{x_i, y_j\} = 0, \quad i \neq j$$

Proposition

Assume that the normalization map preserves the ultra-locality conditions

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad x_i y_j = y_j x_i, \quad i \neq j$$

then (under certain general position conditions) there exists a central non-zero element $q$ such that

$$y_i x_i = q x_i y_i, \quad y'_i x'_i = q x'_i y'_i, \quad i = 1, 2$$

Remark
Ultralocality in the case of the normalization map implies Weyl commutation relations
The full Veblen map \[ \text{[AD, Sergeev 2014]} \]

\[ \begin{align*}
\phi_{AC} &= \phi_{AB}x_1 + \phi_{AD}y_1 \\
\phi_{BC} &= \phi_{AC}x_2 + \phi_{CD}y_2
\end{align*} \quad \text{and} \quad \begin{align*}
\phi_{BC} &= \phi_{AB}\bar{x}_1 + \phi_{BD}\bar{y}_1 \\
\phi_{BD} &= \phi_{AD}\bar{x}_2 + \phi_{CD}\bar{y}_2
\end{align*} \]

Due to the gauge freedom in rescaling \( \phi_{BD} \) the coefficient \( \bar{y}_1 = \lambda \) is free

**Proposition** \[ \text{[AD, Sergeev 2014]} \]

The Veblen map \( V^\lambda : [(x_1, y_1), (x_2, y_2)] \rightarrow [(ar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)] \)

\[ \begin{align*}
\bar{x}_1 &= x_1 x_2, \\
\bar{x}_2 &= y_1 x_2 \lambda^{-1}, \\
\bar{y}_1 &= \lambda \\
\bar{y}_2 &= y_2 \lambda^{-1}
\end{align*} \]

satisfies the pentagon relation \( V^C_{23} \circ V^B_{13} \circ V^A_{12} = V^H_{12} \circ V^G_{23} \) provided \( B = H \) and \( G = CB \)

When the gauge factors are functions of the corresponding arguments then the above relations between the factors become complicated functional equations
**Observation**

In the commutative case the Veblen map $V^\lambda$ with the gauge function $\lambda = \alpha y_1 + \beta x_1 y_2$, where $\alpha, \beta \in k$ are parameters, provides a Poisson automorphism of the field of rational functions $k(x_1, y_1, x_2, y_2)$ equipped with the Poisson structure

$$\{x_i, y_i\} = x_i y_i, \quad \{x_i, x_j\} = \{y_i, y_j\} = \{x_i, y_j\} = 0, \quad i \neq j$$

Moreover, the map satisfies the pentagon relation $V_2^C \circ V_{13}^B \circ V_{12}^A = V_{12}^H \circ V_{23}^G$ provided

$$\alpha_G = \alpha_B \alpha_C, \quad \alpha_H = \alpha_A \alpha_B, \quad \beta_A = \alpha_C \beta_H, \quad \beta_B = \beta_G \beta_H, \quad \beta_C = \alpha_A \beta_G$$

**Proposition**

With the same form of the gauge function the Veblen map preserves the ultralocal Weyl commutation relations

$$y_i x_i = qx_i y_i, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad x_i y_j = y_j x_i, \quad i \neq j$$

Moreover such map satisfies pentagonal condition with the same relations as above between the parameters
The map $R^\lambda = P_{23} \circ V^\lambda_{12} \circ N_{13}$ given explicitly by

\begin{align*}
\tilde{x}_1 &= (x_3 + x_1 y_3)^{-1} x_1 x_2 \\
\tilde{y}_1 &= \alpha y_1 x_1^{-1} x_3 (x_3 + x_1 y_3)^{-1} x_1 + \beta (x_3 + x_1 y_3)^{-1} x_1 y_2 \\
\tilde{x}_2 &= x_3 + x_1 y_3 \\
\tilde{y}_2 &= y_1 y_3 \\
\tilde{x}_3 &= y_1 x_1^{-1} x_3 (x_3 + x_1 y_3)^{-1} x_1 x_2 \tilde{y}_1^{-1} \\
\tilde{y}_3 &= y_2 \tilde{y}_1^{-1}
\end{align*}

preserves ultralocal Weyl commutation relations, and satisfies Zamolodchikov’s tetrahedron condition

\[ R^H_{123} \circ R^G_{145} \circ R^F_{246} \circ R^E_{356} = R^D_{356} \circ R^C_{246} \circ R^B_{145} \circ R^A_{123}, \]

provided the parameters of the gauge functions satisfy

\begin{align*}
\beta_B &= \beta_F \beta_H, & \alpha_C &= \alpha_E \alpha_G, & \alpha_F &= \alpha_B \alpha_D, & \beta_G &= \beta_A \beta_C \\
\alpha_H \beta_F &= \beta_D \alpha_B, & \beta_E \alpha_G &= \beta_C \alpha_A, & \alpha_A \alpha_B &= \alpha_G \alpha_H, & \beta_A \alpha_E &= \alpha_D \beta_H
\end{align*}


A. Doliwa, R. M. Kashaev, **Non-commutative rational pentagon and tetrahedron relations, and Desargues maps**, in preparation