Lump solitons in a higher-order nonlinear equation in $2 + 1$ dimensions

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Burgos. October 2016
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Lump solitons in a higher-order nonlinear equation in 2 + 1 dimensions

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We propose and examine an integrable system of nonlinear equations that generalizes the nonlinear Schrödinger equation to 2 + 1 dimensions. This integrable system of equations is a promising starting point to elaborate more accurate models in nonlinear optics and molecular systems within the continuum limit. The Lax pair for the system is derived after applying the singular manifold method. We also present an iterative procedure to construct the solutions from a seed solution. Solutions with one-, two-, and three-lump solitons are thoroughly discussed.

DOI: 10.1103/PhysRevE.93.062219

I. INTRODUCTION

The cubic nonlinear Schrödinger equation (NLSE) with additional high-order dispersion terms emerges very often in the theoretical description of a number of physical problems in molecular systems, nonlinear optics, and fluid dynamics, to name a few. For instance, the propagation of energy released during adenosine triphosphate hydrolysis, through amide-I vibrations along the hydrogen bonding spine of α-helical proteins, is described by a set of equations which, for dipole-dipole interaction, in the lower order of the continuum approximation is governed by the NLSE. Some years ago, in an attempt to extend the Davydov model for the analysis of molecular excitations [3] that introduce quadrupole-multipole terms in the dispersion of the continuous spectrum, we proposed a generalization of the NLSE [4] to 2 + 1 dimensions. This integrable version appears in the mentioned references as the so-called Lakshmanan-Porsezian-Daniel equation:

\[ i\psi_t + \frac{1}{2} \psi_{xx} + \gamma |\psi|^2 \psi + \frac{1}{2} \psi_{xxxx} + 6\psi_x^2 \psi^* + 4|\psi|^4 \psi + 8|\psi|^2 \psi_x^2 + 2\psi_x \psi_{xx}^2 + 6\psi_x |\psi|^4 = 0, \]

(1)

whose integrability and soliton solutions were studied in Refs. [3] and [4]. More recently, Anikiewicz et al. proposed a further generalization of the NLSE, adding a third-order dispersion term [5]. The integrability of this extended NLSE for some values of the parameters of the equation was confirmed in Ref. [6], where Lax operators were presented. This integrable version appears in the mentioned references as

\[ i\psi_t + \frac{1}{2} \psi_{xx} + \gamma |\psi|^2 \psi + \frac{1}{2} \psi_{xxxx} + 6\psi_x^2 \psi^* + 4|\psi|^4 \psi + 8|\psi|^2 \psi_x^2 + 2\psi_x \psi_{xx}^2 + 6\psi_x |\psi|^4 \]

\[ = \alpha \psi_{xx} + 6\psi_x |\psi|^2. \]

(2)

This equation contains many integrable particular cases such as the standard NLSE (\( \alpha = \gamma = 0 \)) [7], the Hirota equation (\( \gamma = 0 \)) [8], and the Lakshmanan-Porsezian-Daniel equation (\( \alpha = 0 \)) [4]. Soliton solutions and rogue wave for this equation can be found in Refs. [6] and [7] as well as in Refs. [9] and [10].

The relevance of third-order dispersion terms in the context of the self-induced Raman effect have been pointed out by Hesthaven et al. [11]. Moreover, rogue waves in optical fibers can be mathematically described by the NLSE equation and its extensions that take into account third-order dispersion [12].

Models discussed so far are defined in 1 + 1 dimensions as they are aimed at describing the dynamics of the excitations in a single strand of the protein. These models need to include more degrees of freedom and more spatial dimensions to cope with the complex helical geometry of the proteins. To this aim, there exist different generalizations of the NLSE to 2 + 1 dimensions. In particular, we can consider the following system proposed by Calogero in Ref. [13], and then discussed by Zakharov [14], which trivially reduces to the NLSE on the line \( x = y \).

\[ iu_t + u_{xx} + 2u_{yy} = 0, \quad -u_t + u_{xx} + 2u_{yy} = 0, \]

(3)

where \( u = u^* \). This equation has been studied by different authors. A derivation of the Lax pair and Darboux transformations by means of the singular manifold method appears in Ref. [15]. The same method was applied in Ref. [16] to derive rational solutions (lumps) of a different generalization of the NLSE to 2 + 1 dimensions. Notice that the second derivative includes crossed terms \( u_{xy} \) instead of some combination of \( u_{xx} \) and \( u_{yy} \) as appears in many generalizations of NLS. They could be easily recovered through the change of variables \( s = \hat{x} + \hat{y} \) and \( \hat{x} = \hat{x} - \hat{y} \) that yields \( u_{xx} = u_{xx} - u_{yy} \).

Rogue waves in 1 + 1 dimensions [17] as well as lumps in 2 + 1 dimensions [16] are rational solutions with nontrivial behavior. This suggests that rogue waves can appear as a reduction of variables in the lump solutions. As it is well known, lumps are solutions whose meromorphic structure guarantees their stability [18]. This is the main motivation to propose a modified NLSE in 2 + 1 dimensions similarly to the generalization considered in Ref. [15] but including also third- and fourth-order dispersion terms, as in the case of Eq. (2), to be a good candidate for the continuum limit of different discrete models that have been proposed to describe the dynamics of α-helical proteins. The proposed generalization of the set (3) can be cast as a system of equations in the following form:

\[ iu_t + u_{xx} + 2u_{yy} + iu_{xy} - 6uw_{xx} = 0, \]

\[ + \gamma [u_{xx} - 8uw_{xx} - 2u_{xx} - 4uw_{xx} - 6uw_{xx} + 6uw_{xx}^2] = 0, \]

(4a)
The model
Singular Manifold Method
Lumps

high order dispersion terms
The cubic nonlinear Schrödinger equation (NLSE) with additional high order dispersion terms emerges very often in the theoretical description of a number of physical problems in molecular systems.

dipole-dipole interaction
The propagation of energy released during ATP hidrolysis through amide-I vibrations along the hydrogen bonding spine of the alpha-helical proteins is described by a set of equations which, for dipole-dipole interaction, in the lower order of continuum approximation is governed by NLSE.

quadrupole-quadrupole terms
Daniel and Deepmala considered the effects of higher order molecular excitations that introduce quadrupole-quadrupole coefficients.
Lakshmanan-Porsezian-Daniel (1995)

They considered the effects of higher order molecular excitations. The continuum limit yields a generalization of NLSE that includes a fourth-order dispersion term

\[ i\psi_t + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi + \gamma (\psi_{xxxx} + 6\psi_x^2 \psi^* + 4|\psi_x|^2 \psi \]

\[ + 8\psi_{xx} |\psi|^2 + 2\psi_x^2 \psi^2 + 6\psi |\psi|^4) = 0 \]

Ankiewitz et and Akhmediev (2014)

An integrable model adding a third-order dispersion term.

\[ i\psi_t + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi + \gamma (\psi_{xxxx} + 6\psi_x^2 \psi^* + 4|\psi_x|^2 \psi \]

\[ + 8\psi_{xx} |\psi|^2 + 2\psi_x^2 \psi^2 + 6\psi |\psi|^4) \]

\[ = i\alpha (\psi_{xxx} + 6\psi_x |\psi|^2) . \]
Lakshmanan-Porsezian-Daniel (1995)

They considered the effects of higher order molecular excitations. The continuum limit yields a generalization of NLSE that includes a fourth-order dispersion term

\[ i\psi_t + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi + \gamma (\psi_{xxxx} + 6\psi_x^2 \psi^* + 4|\psi_x|^2 \psi \\
+ 8\psi_{xx}|\psi|^2 + 2\psi^* \psi_x^2 + 6\psi |\psi|^4) = 0 , \]

Ankiewitz et and Akhmediev (2014)

An integrable model adding a third-order dispersion term.

\[ i\psi_t + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi + \gamma (\psi_{xxxx} + 6\psi_x^2 \psi^* + 4|\psi_x|^2 \psi \\
+ 8\psi_{xx}|\psi|^2 + 2\psi^* \psi_x^2 + 6\psi |\psi|^4) = i\alpha (\psi_{xxx} + 6\psi_x |\psi|^2) . \]
Generalizations of NLS to 2+1

Calogero (1975)

\[ iu_t + u_{xy} + 2um_y = 0 , \]
\[ -iw_t + w_{xy} + 2wm_y = 0 , \]
\[ m_x + uw = 0 , \]

Rogue waves and lumps

Rogue waves in \(1+1\) dimensions as well as lumps in \(2+1\) dimensions are rational solutions with non trivial behavior. This suggests that rogue waves can appear as a reduction of variables in the lump solutions. This is the main motivation to propose a modified NLSE in \(2+1\) dimensions but including also third- and fourth-order dispersion terms.
Our model (2016)

\[ iu_t + u_{xy} + 2um_y + i\alpha (u_{xxx} - 6uwu_x) \]
\[ + \gamma (u_{xxxx} - 8uwu_{xx} - 2u^2w_{xx} \]
\[ - 4uu_xw_x - 6wu_x^2 + 6u^3w^2) = 0 , \]

\[ -iw_t + w_{xy} + 2wm_y - i\alpha (w_{xxx} - 6uww_x) \]
\[ + \gamma (w_{xxxx} - 8uvw_{xx} - 2w^2u_{xx} \]
\[ - 4wu_xw_x - 6uw_x^2 + 6u^2w^3) = 0 , \]

\[ m_x + uw = 0 . \]
References

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We propose and examine an integrable system of nonlinear equations that generalizes the nonlinear Schrödinger equation to $2 + 1$ dimensions.

This integrable system of equations is a promising starting point to elaborate more accurate models in nonlinear optics and molecular systems within the continuum limit.

The Lax pair for the system is derived after applying the singular manifold method.

We also present an iterative procedure to construct the solutions from a seed solution.

Solutions with one, two and three lump solitons are thoroughly discussed.
Painlevé expansion

\[ u = \sum_{j=0}^{\infty} a_j(x, y, t) \left[ \phi(x, y, t) \right]^{j-1}, \]
\[ w = \sum_{j=0}^{\infty} b_j(x, y, t) \left[ \phi(x, y, t) \right]^{j-1}, \]
\[ m = \sum_{j=0}^{\infty} m_j(x, y, t) \left[ \phi(x, y, t) \right]^{j-1}, \]

where \( \phi(x, y, t) \) is an arbitrary function. This means that all solutions are single-valued around the singularity manifold \( \phi(x, y, t) = 0 \).
There exists a resonance in $j = 0$. A function $g_1$ is introduced to give account of this arbitrariness

$$u^{[1]} = u^{[0]} + \frac{g_1\phi_{1,x}}{\phi_1},$$

$$w^{[1]} = w^{[0]} + \frac{\phi_{1,x}}{g_1\phi_1},$$

$$m^{[1]} = m^{[0]} + \frac{\phi_{1,x}}{\phi_1},$$

Solutions obtained through this expansion in the singular manifold $\phi_1$ have been denoted as $(u^{[1]}, w^{[1]}, m^{[1]})$. It implies an iterative method of construction of solutions where the superindex $[0]$ denotes a seed solution and $[1]$ the iterated one.
Substitution of the truncated expansion leads to three polynomials in powers of $\phi_1$.

By imposing that each coefficient vanishes, we obtain a set of equations: the singular manifold equations that relates the singular manifold $\phi_1$ with the seed solution $(u^{[0]}, w^{[0]}, m^{[0]})$.

The process of obtaining these equations requires some tedious but straightforward calculation that we have performed with the aid of the symbolic calculus package MAPLE.
Lax pair: The singular manifold equations can be linearized

\[
\begin{align*}
\psi_{1,x} + u^{[0]} \varphi_1 + i \lambda_1 \psi_1 &= 0, \\
\psi_{1,t} &= 2 \lambda_1 \psi_{1,y} + \lambda_{1,y} \psi_1 + i \left( m^{[0]}_y \psi_1 - u^{[0]}_y \varphi_1 \right) \\
&\quad + (\alpha - 2 \lambda_1 \gamma) F \left[ \psi_1, \varphi_1, \lambda_1, u^{[0]}, w^{[0]} \right] + i \gamma G \left[ \psi_1, \varphi_1, u^{[0]}, w^{[0]} \right] \\
\varphi_{1,x} + w^{[0]} \psi_1 - i \lambda_1 \varphi_1 &= 0, \\
\varphi_{1,t} &= 2 \lambda_1 \varphi_{1,y} + \lambda_{1,y} \varphi_1 - i \left( m^{[0]}_y \varphi_1 - w^{[0]}_y \psi_1 \right) \\
&\quad + (\alpha - 2 \lambda_1 \gamma) F \left[ \varphi_1, \psi_1, \lambda_1, w^{[0]}, u^{[0]} \right] - i \gamma G \left[ \varphi_1, \psi_1, w^{[0]}, u^{[0]} \right] \\
F [\psi, \varphi, \lambda, u, w] &= 3(uw + \lambda^2)\psi_x - \psi_{xxx} - 3u_x \varphi_x, \\
G [\psi, \varphi, u, w] &= \left( 3u^2w^2 + u_x w_x - uw_{xx} - wu_{xx} \right) \psi + (6uwu_x - u_{xxx}) \varphi.
\end{align*}
\]
Non-isospectral Lax pair

\[ \lambda_t - 2\lambda \lambda_y = 0 \]
The model
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Singular manifold and eigenfunctions

\[
\begin{align*}
\phi_{1,x} &= \psi_1 \varphi_1, \\
\phi_{1,t} &= 2\lambda_1 \phi_{1,y} + i (\varphi_1 \psi_{1,y} - \psi_1 \varphi_{1,y}) \\
&\quad + (\alpha - 2\lambda_1 \gamma) J [\psi_1, \varphi_1, \lambda_1] + i \gamma K [\psi_1, \varphi_1, u^{[0]}, w^{[0]}].
\end{align*}
\]

where we have defined

\[
\begin{align*}
J [\psi, \varphi, \lambda] &= 4\psi_x \varphi_x + 6\lambda^2 \psi \varphi - \psi \varphi_{xx} - \varphi \psi_{xx}, \\
K [\psi, \varphi, u, w] &= \varphi \psi_{xx} - \psi \varphi_{xx} + 3(\psi_x \varphi_{xx} - \varphi_x \psi_{xx}) + 6uw (\psi \varphi_x - \varphi \psi_x).
\end{align*}
\]
These Lax pairs can be considered as nonlinear equations between the fields and the eigenfunctions.
These Lax pairs can be considered as nonlinear equations between the fields and the eigenfunctions.

The crucial point here is to consider the Lax pair as a set of nonlinear equations for the fields and eigenfunctions together:

\[ \psi_{1,2} = \psi_2 - \psi_1 \frac{\Delta_{1,2}}{\phi_1}, \]

\[ \phi_{1,2} = \phi_2 - \phi_1 \frac{\Delta_{1,2}}{\phi_1}, \]

where \((\psi_i, \phi_i)\) are eigenfunctions of the Lax pair for \((u^{[0]}, w^{[0]})\) with eigenvalue \(\lambda_j\) \((j = 1, 2)\).

\[ \Delta_{1,2} = \frac{i}{2} \frac{\varphi_1 \psi_2 - \varphi_2 \psi_1}{\lambda_2 - \lambda_1}. \]
Summarizing

\[ u^{[2]} = u^{[0]} + \frac{1}{\tau_{1,2}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} \Delta_{2,2} & -\Delta_{1,2} \\ -\Delta_{1,2} & \Delta_{1,1} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]

\[ w^{[2]} = w^{[0]} + \frac{1}{\tau_{1,2}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \begin{pmatrix} \Delta_{2,2} & -\Delta_{1,2} \\ -\Delta_{1,2} & \Delta_{1,1} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \]

\[ m^{[2]} = m^{[0]} + \frac{(\tau_{1,2})_x}{\tau_{1,2}}, \]

\[ \tau_{1,2} = \phi_{1,2}\phi_1 = \phi_2\phi_1 - (\Delta_{1,2})^2 = \det(\Delta), \]
Solve the Lax pair in order to obtain the eigenfunctions \((\phi_i, \varphi_i)\) for a given seed solution \((u[0], w[0])\) and spectral parameter \(\lambda_i\).

These eigenfunctions allow us to construct the \(n \times n\) matrix \(\Delta\)

\[
\Delta_{j,k} = \frac{i}{2} \frac{\varphi_j \psi_k - \varphi_k \psi_j}{\lambda_k - \lambda_j}, \quad j, k : 1...n.
\]

The \(\tau\) functions of order \(n\) can be obtained as the determinant of the matrix \(\Delta\).

\[
\tau_{1,2,...,n} = \phi_1 \phi_{1,2} \cdots \phi_{1,2,...,n} = \det(\Delta),
\]

The probability density \(u[n]w[n] = -m_x^{[n]}\) can be easily obtained through

\[
m_x^{[n]} = m_x^{[0]} + \left( \frac{\tau_{1,2,...,n}}{\tau_{1,2,...,n}} \right)_x.
\]
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Lumps: $u^{[0]} = ij_0$, $w^{[0]} = -ij_0$, $m^{[0]} = j_0^2x - 3\gamma j_0^4y$,

where $j_0$ is an arbitrary constant. We can obtain rational solutions of the Lax pair as

\[
\psi_1 = j_1 + j_0(j_2 - j_1) \left\{ x + b_1j_0^2y + 2 \left[ -3\alpha - ij_0^2(b_1 + 6\gamma) \right] j_0 t \right\},
\]

\[
\varphi_1 = j_2 + j_0(j_2 - j_1) \left\{ x + b_1j_0^2y + 2 \left[ -3\alpha + ij_0^2(b_1 + 6\gamma) \right] j_0 t \right\},
\]

\[
\psi_2 = j_2 + j_0(j_2 - j_1) \left\{ x + b_1j_0^2y + 2 \left[ -3\alpha - ij_0^2(b_1 + 6\gamma) \right] j_0 t \right\},
\]

\[
\varphi_2 = j_1 + j_0(j_2 - j_1) \left\{ x + b_1j_0^2y + 2 \left[ -3\alpha + ij_0^2(b_1 + 6\gamma) \right] j_0 t \right\},
\]

where $\lambda_1 = -\lambda_2 = ij_0$

and $j_1$, $j_2$ and $b_1$ are three arbitrary constants. This particular choice of $\lambda_2$ as the complex conjugate of $\lambda_1$ has been made in order to have $\phi_2$ as the complex conjugate of $\phi_1$. Our goal is to get a real $\tau$-function without zeroes and therefore solutions without singularities.
\[ \tau_{1,2} = A^2 + B^2 + C^2, \quad Z_t - 2ij_0Z_y = 0 \]

\[ A = 12\alpha h_1 j_0^7 t^2 \left\{ 2\delta X + 3y \left( 3\alpha^2 - j_0^2 \delta^2 \right) \right\} - 2h_1 j_0^5 \delta t \left[ X^2 + j_0^2 y^2 \left( 27\alpha^2 - j_0^2 \delta^2 \right) \right] + 2h_2 j_0^4 t \left[ \delta X + y \left( 9\alpha^2 - j_0^2 \delta^2 \right) \right] - h_1 \alpha j_0 y \left[ j_0^4 y^2 \left( 9\alpha^2 - 3j_0^2 \delta^2 \right) + 2 \right] - 3h_2 \alpha \delta j_0^4 y^2 + 3\alpha j_0 j_1 j_2 y + \text{Re}[Z], \]

\[ B = 4h_1 j_0^6 t^2 \left\{ X \left( 9\alpha^2 - 3j_0^2 \delta^2 \right) - y \delta j_0^2 \left( 27\alpha^2 - j_0^2 \delta^2 \right) \right\} - 6h_1 \alpha j_0^4 t \left[ X^2 + j_0^2 y^2 \left( 3\alpha^2 - j_0^2 \delta^2 \right) \right] + 6h_2 \alpha j_0^3 t \left( X - 2\delta j_0 y \right) + \frac{1}{6} h_1 j_0^2 \left[ 2X^3 + 2\delta j_0^4 y^3 \left( 27\alpha^2 - j_0^2 \delta^2 \right) \right] + \frac{1}{6} h_1 j_0^2 \left[ y \left( 7\delta - 4b_1 \right) \right] - \frac{1}{2} h_2 j_0 \left[ X^2 + j_0^2 y^2 \left( 9\alpha^2 - 3j_0^2 \delta^2 \right) \right] + j_1 j_2 (X - j_0^2 \delta y) + \text{Im}[Z], \]

\[ C = \frac{h_1}{4j_0} \left[ 2j_0^2 (6j_0^2 \alpha t - X)^2 + 8j_0^8 \delta^2 t^2 + 1 \right] + \frac{h_2}{2} (6j_0^2 \alpha t - X) + \frac{j_1 j_2}{j_0}, \]

and we have introduced the notation

\[ X = x + b_1 j_0^2 y, \]

\[ h_1 = (j_1 - j_2)^2, \quad h_2 = j_1^2 - j_2^2, \]

\[ \delta = 6\gamma + b_1. \]
$j_1 = j_2, \quad Z = 0$

$$\tau_{1,2} = (3\alpha j_0 y)^2 + (X - j_0^2 \delta y)^2 + \frac{1}{j_0^2}$$
\( j_1 = j_2, \ Z = a_1(y + 2ij_0 t)^2 \) with \( a_1 \) real

\[
\tau = \left(3j_0 \alpha y + a_1 y^2 - 4a_1 j_0^2 t^2\right)^2 + \left(X - j_0^2 y \delta + 4a_1 y j_0 t\right)^2 + \frac{1}{4j_0^2}
\]
\[ j_1 = j_2, \quad Z = a_1(y + 2ij_0t)^3 \text{ with } a_1 \text{ real} \]

\[
\tau = \left( 3j_0 \alpha y + a_1 y^3 - 12a_1 y j_0^2 t^2 \right)^2 + \\
\left( X - j_0^2 y \delta + 6a_1 y^2 j_0 t - 8a_1 j_0^3 t^3 \right)^2 + \frac{1}{4j_0^2}
\]
$j_1 = j_2, \ Z = ia_1 (y + 2ij_0 t)^2$ with $a_1$ real
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\[ j_1 = j_2, \quad Z = i a_1 (y + 2ij_0 t)^3 \] with \( a_1 \) real
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\[ j_2 = -j_1 \]
References


