A shortcut to the Kovalevskaya curve

\[ w^2 = \left[ s \left( (s - H)^2 + 1 - l_2/4 \right) - l_1 \right] \left( (s - H)^2 - l_2/4 \right) \]

Yuri Fëdorov, UPC, Barcelona
with participation of Luis Garcia Naranjo (UNAM, México)

Nonlinear Integrable Systems
The Kolvalevskaya top (1891)
(equations of motion and integrals)

The quadratures
\[ ds_1 \sqrt{P_5(s_1)} + ds_2 \sqrt{P_5(s_2)} = 0 \]
\[ s_1 ds_1 \sqrt{P_5(s_1)} + s_2 ds_2 \sqrt{P_5(s_2)} = \sqrt{2} dt \]

Kovalevskaya curve \( w^2 = R_5(s) \)

Lax pair with
genus 3 spectral curve

BRS (1991)

Explicit solution

change of variables

shortcut ?
The classical Kovalevskaya top:  
Equations of motion, first integrals, Lax pair

\[
\begin{align*}
\frac{dl}{dt} &= [l, \omega] + [c, g], \\
\frac{dg}{dt} &= [g, \omega], \\
\omega &= (\omega_1, \omega_2, \omega_3) = (l_1, l_2, 2l_3), \\
g &= (g_1, g_2, g_3), \\
c &= (1, 0, 0),
\end{align*}
\]

\[
H = \frac{1}{2}(l_1^2 + l_2^2 + 2l_3^2) - g_1,
\]

and there are two additional integrals of motion

\[
I_1 = (l, g)^2, \\
I_2 = (l_1^2 - l_2^2 + 2g_1)^2 + 4(l_1l_2 + g_2)^2.
\]

The equations of motion admit a Lax representation

\[
\frac{dL}{dt} = [L, M]
\]

with Lax matrix

\[
L(\lambda) = \\
\begin{bmatrix}
\frac{g_1}{\lambda} & \frac{g_2}{\lambda} & -l_2 + \frac{g_3}{\lambda} & -l_1 \\
\frac{g_2}{\lambda} & -\frac{g_1}{\lambda} & l_1 & -l_2 - \frac{g_3}{\lambda} \\
l_2 + \frac{g_3}{\lambda} & -l_1 & -2\lambda - \frac{g_1}{\lambda} & -2l_3 + \frac{g_2}{\lambda} \\
l_1 & l_2 - \frac{g_3}{\lambda} & 2l_3 + \frac{g_2}{\lambda} & 2\lambda + \frac{g_1}{\lambda}
\end{bmatrix},
\]

The spectral curve $C : |L(\lambda) - \mu I| = 0$, after factorization by a trivial involution, can be written in the following equivalent form

$$y^2 = (x^2 - 2Hx + 2)x + x\sqrt{-4I_1^2x^3 + (4H^2 - I_2 + 4)x^2 - 8Hx + 4}.$$ 

It admits involution $\sigma : (x, y) \rightarrow (x, -y)$ and is 2-fold covering of the elliptic curve

$$E : z^2 = -4I_1^2x^3 + (4H^2 - I_2 + 4)x^2 - 8Hx + 4,$$

the covering $C \rightarrow E$ ramified at 4 points on $E$. So $\text{genus}(C) = 3$.

**Theorem (B., R., S-T.)**

*The generic 2-dim. complex invariant tori of the system are (open subsets of) $\text{Prym}(C/E) \subset \text{Jac}(C)$*
The classical separation of variables by Kovalevskaya (1889)

The components of $l, g$ are certain (complicated !) algebraic functions of $s_1, w_1, s_2, w_2, H, l_2, L$ such that

$$w_j^2 = P_5(s_j), \quad j = 1, 2, \quad P_5(s) = (s^2 - l_2)[(s^2 - l_2 + 4)(s - 2H) + 8l_1^2],$$

$$\frac{ds_1}{\sqrt{P_5(s_1)}} + \frac{ds_2}{\sqrt{P_5(s_2)}} = 0$$

$$\frac{s_1 ds_1}{\sqrt{P_5(s_1)}} + \frac{s_2 ds_2}{\sqrt{P_5(s_2)}} = \sqrt{2} \, dt$$

The Kovalevskaya genus 2 curve of separation:

$$\mathcal{K} : \ w^2 = (s^2 - l_2)[(s^2 - l_2 + 4)(s - 2H) + 8l_1^2]$$

- So, the 2-dim. Abelian tori $\text{Jac}(\mathcal{K})$ and $\text{Prym}(C/E)$ must be isogeneous (i.e., algebraically related).

- How the genus 3 spectral curve $C$ and the curve $\mathcal{K}$ are related ?
The puzzle of 6 Jacobians and 2 Pryms

Following
and
V. Enolski, Yu. F. *Algebraic description of Jacobians isogeneous to Prym varieties with polarization (1,2).*

E. Horozov & P. van Moerbeke:
The Kovalevskaya curve $\mathcal{K}$ is equivalent to one of the curves $\tilde{\Gamma}_i$. 
W. Barth’s construction (1984): pencils of dual curves

\[ K = \text{Prym}(C/E) \cap \Theta_C \]

\[ C = \text{Prym}(K/\mathcal{E}) \cap \Theta_K \]

\[ \text{Prym}(C/E) \subset \text{Jac}(C) \]

\[ \text{Prym}(K/\mathcal{E}) \subset \text{Jac}(K) \]

Pencil of covers \( K_\lambda \to E_\lambda \) giving the same \( \text{Prym}(K/\mathcal{E}) \)

Pencil of covers \( C_\lambda \to E_\lambda \) giving the same \( \text{Prym}(C/E) \)

\[ g = 3 \quad g = 1 \]

\[ \lambda \in \mathcal{P} \]
Theorem [Horozov & van Moerbeke (1989)]

\[ \begin{align*}
C & \quad C_{\lambda} \\
H_1 & \quad S_1 & \quad H_2 & \quad S_2 & \quad S_3 & \quad C & \quad H_3 \\
2 : 1 & \quad \Leftrightarrow & \quad 2 : 1 & \quad \Leftrightarrow & \quad 2 : 1 & \quad \Leftrightarrow & \quad 2 : 1 \\
\tilde{\Gamma}_1 & \quad \tilde{\Gamma}_2 & \quad \Gamma_1 & \quad \tilde{\Gamma}_3 & \quad \Gamma_2 & \quad \tilde{\Gamma}_3 & \quad \Gamma_3 \\
S_1 & \quad \hat{S}_1 & \quad \hat{H}_1 & \quad \hat{H}_2 & \quad \hat{S}_3 & \quad \hat{H}_3 \\
\Leftrightarrow & \quad \Leftrightarrow & \quad 2 : 1 & \quad 2 : 1 & \quad \Leftrightarrow & \quad 2 : 1 \\
\tilde{\Gamma}_1 & \quad \tilde{\Gamma}_2 & \quad \Gamma_1 & \quad \tilde{\Gamma}_3 & \quad \Gamma_2 & \quad \tilde{\Gamma}_3 & \quad \Gamma_3 \\
\uparrow & \quad \text{Jac}(\tilde{\Gamma}_1) & \quad \text{Jac}(\tilde{\Gamma}_2) & \quad \text{Jac}(\tilde{\Gamma}_3) & \quad \downarrow & \quad \text{Jac}(\Gamma_1) & \quad \text{Jac}(\Gamma_2) & \quad \text{Jac}(\Gamma_3)
\end{align*} \]
Any genus 3 curve which is also a 2-fold covering of an elliptic curve, can be written in form

\[ C : \quad w^2 = g_3(x) + (\alpha x + \beta)\sqrt{\Phi(x)}, \quad \Phi(x) := (x - c_1)(x - c_2)(x - c_3) \]

such that \( g_3^2(x) - (\alpha x + \beta)^2\Phi(x) = (x - x_1) \cdots (x - x_4) \rho^2(x) \)

for a certain linear \( \rho(x) \).

Consider another genus 3 curve

\[ K : \quad w^2 = g_3(x) + \rho(x)\sqrt{\psi(x)} \]

which is a 2-fold covering of \( \mathcal{E} = \{y^2 = \psi(x)\} \).

Theorem (L. Heine (1983), S. Pantasis (1986))

Prym\((C/E)\) and Prym\((K/E)\) are dual in the sense of definition.
The main observation

The dual to the genus 3 spectral curve of the Lax pair of Reiman and Semenov-Tian-Shanski

\[ C : w^2 = (x^2 - 2Hx + 2)x + x\sqrt{-4l_1^2x^3 + (4H^2 - l_2 + 4)x^2 - 8Hx + 4} \]

is

\[ w^2 = (x^2 - 2Hx + 2)x + x^2 \sqrt{x^2 + 4(l_1^2 - H)x + l_2} \]

The latter is *singular*, and its regularization is birationally equivalent to the genus 2 curve of Kovalevskaya

\[ \mathcal{K} : w^2 = (s^2 - l_2)[(s^2 - l_2 + 4)(s - 2H) + 8l_1^2] \]

**Corollary.** The curve \( \mathcal{K} \) is equivalent to one of the genus 2 curves \( \tilde{\Gamma}_\alpha \), and \( \text{Jac}(\mathcal{K}) \) is a 2-fold covering of \( \text{Prym}(C/E) \), the complex invariant manifold of the system.

- In the particular case \( l_1 = 0 \) (zero area integral) a similar observation has been made by F. Leprovost & A. Markuschevich (1999)