Classification of five-point differential-difference equations

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work done in collaboration with
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Nonlinear Integrable Systems - Workshop to honor Orlando Ragnisco

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• I first met Orlando in 1975.

• Our collaboration started then with our first paper in common on work in Nuclear Physics on *Perturbative approach to some one-dimensional many-body problems* in 1976.

• Our last work done in collaboration is on *On Fourier integral transforms for q-Fibonacci and q-Lucas polynomials* in 2012.

• In this period we produced 35 papers together.

• We organized two conferences together.

• We edited a book together.
Figure 1: Rome La Sapienza Nonlinear Group. Photo taken in the occasion of Srinivasa Rao visit in 1996.
1. Introduction

2. Theoretical consequences of the existence of generalized symmetries.
   (a) Class I integrability conditions.
   (b) Criteria for checking the integrability conditions.

3. Lists of integrable differential – difference equations of Class I.

4. Generalized symmetries of key equations.

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1 Introduction

The generalized symmetry method uses the existence of generalized symmetries as an integrability criterion and allows one to classify integrable equations of a certain class. Using this method, the classification problem has been solved for some important classes of Partial Differential Equations (PDEs) (Mikhailov, Shabat, Sokolov, Yamilov), of Differential-Difference Equations (D\(\Delta\)Es) (Adler Shabat, Yamilov), and of Partial Difference Equations (P\(\Delta\)Es) (Gariollow, L., Yamilov).

This is not the only integrability criterion introduced to produce integrable P\(\Delta\)Es. Using the Compatibility Around the Cube (CAC) technique introduced by Adler, Bobenko and Suris (ABS), one obtains classes of integrable equations on a quad graph. All equations obtained by ABS and their extensions have generalized symmetries which are integrable D\(\Delta\)Es, belonging to the classification presented by L. and Yamilov and given, in general, by D\(\Delta\)Es defined on three-point lattices.
An extension of the classification of the integrable PΔEs defined on a square is very difficult to perform. An alternative that seems more easy to perform is to **classify integrable five-point DΔEs**

\[ \dot{u}_n = \Psi(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}). \]  

(1.1)

The **integrable PΔEs are then obtained as Bäcklund transformations of these DΔEs.**

An example in this class is the Ito-Narita-Bogoyavlensky (INB) equation:

\[ \dot{u}_n = u_n(u_{n+2} + u_{n+1} - u_{n-1} - u_{n-2}). \]  

(1.2)

Volterra type equations

\[ \dot{u}_n = \Phi(u_{n+1}, u_n, u_{n-1}) \]  

(1.3)

have been completely classified. The **classification of five-point lattice equations of the form** (1.1) **will contain equations coming from the classification of Volterra equations** (1.3). For example

\[ \dot{u}_n = \Phi(u_{n+2}, u_n, u_{n-2}). \]  

(1.4)
Eq. (1.4) is just a three-point lattice equation equivalent to (1.3). A second case is when we consider generalized symmetries of (1.3). For INB the simplest generalized symmetry has the form:

\[ u_{n,\tau} = G(u_{n+4}, u_{n+3}, u_{n+2}, u_{n+1}, u_{n}, u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}). \]  

We consider here equations of the form

\[ \dot{u}_n = A(u_{n+1}, u_n, u_{n-1})u_{n+2} + B(u_{n+1}, u_n, u_{n-1})u_{n-2} + C(u_{n+1}, u_n, u_{n-1}). \]  

The problem naturally splits into cases depending on the form of the functions \( A \) and \( B \).
In this article we study the case when the autonomous DΔEs (1.6) is such that

\[ A = a_n \neq \alpha(u_{n+1}, u_n)\alpha(u_n, u_{n-1}), \quad B = b_n \neq \beta(u_{n+1}, u_n)\beta(u_n, u_{n-1}) \]

We call this class the \textit{Class I}.

\textbf{Theorem 1} \textit{The request that conditions (1.7) are satisfied is equivalent to}

\[ \frac{\partial}{\partial u_n} \frac{a_{n+1}a_{n-1}}{a_n} \neq 0, \quad \frac{\partial}{\partial u_n} \frac{b_{n+1}b_{n-1}}{b_n} \neq 0, \quad (1.8) \]
2  Theorical consequences of the existence of a generalized symmetry

Notation:

\[ \dot{u}_n = a_n u_{n+2} + b_n u_{n-2} + c_n \equiv f_n. \]  \hfill (2.1)

\[ u_{n,\tau} = g_n(u_{n+4}, u_{n+3}, u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}), \]  \hfill (2.2)

The compatibility condition for (2.1) and (2.2) is

\[ u_{n,\tau,t} - u_{n,t,\tau} \equiv D_t g_n - D_\tau f_n = 0. \]  \hfill (2.3)

\( D_t \) and \( D_\tau \) are the operator of total differentiation with respect to \( t \) and \( \tau \):

\[ D_t = \sum_{k \in \mathbb{Z}} f_k \frac{\partial}{\partial u_k}, \quad D_\tau = \sum_{k \in \mathbb{Z}} g_k \frac{\partial}{\partial u_k}. \]  \hfill (2.4)

We assume as independent variables the functions

\[ u_0, u_1, u_{-1}, u_2, u_{-2}, u_3, u_{-3}, \ldots. \]  \hfill (2.5)
Eq. (2.3) depends on variables \( u_{-6}, u_{-5}, \ldots, u_5, u_6 \) it is an overdetemined equation for the unknown function \( g_0 \), given \( f_0 \).

We can calculate \( g_0 \) step by step, obtaining conditions for the function \( f_0 \).

Differentiating (2.3) with respect to \( u_{\pm 6} \) we obtain:

\[
\frac{\partial g_0}{\partial u_4} = a_0 a_2, \quad \frac{\partial g_0}{\partial u_{-4}} = \nu b_0 b_{-2}. \tag{2.6}
\]

Differentiating (2.3) with respect to \( u_{\pm 5} \) and defining

\[
h_0^+ = \frac{\partial g_0}{\partial u_3} - a_1 \frac{\partial f_0}{\partial u_1} - a_0 \frac{\partial f_2}{\partial u_3}, \quad h_0^- = \frac{\partial g_0}{\partial u_{-3}} - \nu b_{-1} \frac{\partial f_0}{\partial u_{-1}} - \nu b_0 \frac{\partial f_{-2}}{\partial u_{-3}}.
\]

**Lemma 1** If \( h_0^\pm \neq 0 \), then there exists \( \hat{\alpha}_n = \alpha(u_n, u_{n-1}) \) and \( \hat{\beta}_n = \beta(u_{n+1}, u_n) \), such that \( a_0 = \hat{\alpha}_1 \hat{\alpha}_0 \) and \( b_0 = \hat{\beta}_0 \hat{\beta}_{-1} \).
(a) \( a_0 \neq \hat{\alpha}_1 \hat{\alpha}_0 \) and \( b_0 \neq \hat{\beta}_0 \hat{\beta}_{-1} \) then \( h_0^\pm = 0 \) due to Lemma 1.

(b) \( a_0 = \hat{\alpha}_1 \hat{\alpha}_0 \) and \( b_0 = \hat{\beta}_0 \hat{\beta}_{-1} \) then \( h_0^+ = \mu^+ \hat{\alpha}_0 \hat{\alpha}_1 \hat{\alpha}_2 \) and \( h_0^- = \mu^- \hat{\beta}_0 \hat{\beta}_{-1} \hat{\beta}_{-2} \), \( \mu^\pm \) constants.

In both cases we get \( \frac{\partial g_0}{\partial u_{\pm 3}} \).

The **Lemma** provides a natural frame for splitting the calculation of \( g_0 \) into two cases i.e. define the **Class I**.
2.1 Class I integrability conditions

When $h_0^+ = h_0^- = 0$

$$\frac{\partial g_0}{\partial u_4} = a_0 a_2, \quad \frac{\partial g_0}{\partial u_3} = a_1 \frac{\partial f_0}{\partial u_1} + a_0 \frac{\partial f_2}{\partial u_3}, \quad (2.7)$$

$$\frac{\partial g_0}{\partial u_{-4}} = \nu b_0 b_{-2} \quad \frac{\partial g_0}{\partial u_{-3}} = \nu b_{-1} \frac{\partial f_0}{\partial u_{-1}} + \nu b_0 \frac{\partial f_{-2}}{\partial u_{-3}}. \quad (2.8)$$

Differentiating the compatibility conditions with respect to $u_4$ and $u_{-4}$ and introducing the functions:

$$q_0^+ = \frac{1}{a_0} \frac{\partial g_0}{\partial u_2} - D_t \log a_0 - \frac{\partial f_0}{\partial u_0} - \frac{\partial f_2}{\partial u_2} - \frac{1}{a_0} \frac{\partial f_0}{\partial u_1} \frac{\partial f_1}{\partial u_2}, \quad (2.9)$$

$$q_0^- = \frac{1}{\nu b_0} \frac{\partial g_0}{\partial u_{-2}} - D_t \log b_0 - \frac{\partial f_0}{\partial u_0} - \frac{\partial f_{-2}}{\partial u_{-2}} - \frac{1}{b_0} \frac{\partial f_0}{\partial u_{-1}} \frac{\partial f_{-1}}{\partial u_{-2}}, \quad (2.10)$$

we obtain, up to a common factor $a_0 a_2$ and $\nu b_0 b_{-2}$, the relations:

$$2D_t \log a_0 = q_2^+ - q_0^+, \quad (2.11)$$

$$2D_t \log b_0 = q_{-2}^- - q_0^-. \quad (2.12)$$
The relations (2.11) and (2.12) have the form of conservation laws and are necessary conditions for the integrability, formulated in terms of equation (2.1) only.

If, for a given equation (2.1), conditions (2.11) and (2.12) are satisfied and the functions $q_n^\pm$ are known, then partial derivatives $\frac{\partial g_0}{\partial u_2}, \frac{\partial g_0}{\partial u_{-2}}$ can be found from (2.9, 2.10). In this case the right hand side of symmetry (2.2) is defined up to one unknown function of 3 variables:

$$\psi(u_{n+1}, u_n, u_{n-1}). \quad (2.13)$$

This function can be found directly from the compatibility condition (2.3).

In this way we can carry out the classification of the equations of Class I. On the first stage we use the integrability conditions (2.11, 2.12). Then we define the symmetry up to function (2.13) and try to find it from the compatibility condition.
2.2 Criteria for checking the integrability conditions

Let us present some criteria for checking the integrability conditions (2.11, 2.12).
Let us define

\[ \varphi = \varphi(u_{m_1}, u_{m_1-1}, \ldots, u_{m_2}), \quad m_1 \geq m_2. \]  

(2.14)

the formal variational derivative:

\[ \frac{\delta \varphi}{\delta u_0} = \sum_{k=m_2}^{m_1} T^{-k} \frac{\partial \varphi}{\partial u_k} = \frac{\partial}{\partial u_0} \sum_{k=m_2}^{m_1} T^{-k} \varphi, \]  

(2.15)

and its adjoint:

\[ \frac{\bar{\delta} \varphi}{\bar{\delta} u_0} = \sum_{k=m_2}^{m_1} (-1)^k T^{-k} \frac{\partial \varphi}{\partial u_k} = \frac{\partial}{\partial u_0} \sum_{k=m_2}^{m_1} (-1)^k T^{-k} \varphi, \]  

(2.16)
Theorem 2  The conditions
\[
\frac{\delta}{\delta u_0} D_t \log a_0 = 0, \quad \frac{\bar{\delta}}{\delta u_0} D_t \log a_0 = 0 \quad (2.17)
\]
imply
\[
2D_t \log a_0 = \kappa^+ + (T^2 - 1)q_0^+, \quad \kappa^+ \in \mathbb{C}. \quad (2.18)
\]
and viceversa

Theorem 3  The conditions
\[
\frac{\delta}{\delta u_0} D_t \log b_0 = 0, \quad \frac{\bar{\delta}}{\delta u_0} D_t \log b_0 = 0 \quad (2.19)
\]
imply
\[
2D_t \log b_0 = \kappa^- + (T^{-2} - 1)q_0^-, \quad \kappa^- \in \mathbb{C}. \quad (2.20)
\]
and viceversa
We are led to

\[
\frac{\partial}{\partial u_0} D_t \log(a_{-2}a_0a_2) = 0, \quad (2.21a)
\]
\[
\frac{\partial}{\partial u_0} D_t \log(a_{-3}a_{-1}a_1a_3) = 0, \quad (2.21b)
\]
\[
\frac{\partial}{\partial u_0} D_t \log(b_{-2}b_0b_2) = 0, \quad (2.22a)
\]
\[
\frac{\partial}{\partial u_0} D_t \log(b_{-3}b_{-1}b_1b_3) = 0. \quad (2.22b)
\]

These are explicit and simple conditions for checking the integrability conditions (2.11) and (2.12).
We have explicit formulae for the partial derivatives

\[
\frac{\partial g_0}{\partial u_4}, \quad \frac{\partial g_0}{\partial u_{-4}}, \quad \frac{\partial g_0}{\partial u_3}, \quad \frac{\partial g_0}{\partial u_{-3}}
\]

(2.23)

and implicit definitions for

\[
\frac{\partial g_0}{\partial u_2}, \quad \frac{\partial g_0}{\partial u_{-2}}.
\]

(2.24)

The compatibility of the partial derivatives led to

\[
(\nu + 1) \frac{\partial b_0}{\partial u_1} = 0, \quad (\nu + 1) \frac{\partial a_0}{\partial u_{-1}} = 0, \quad (\nu + 1) \frac{\partial}{\partial u_0} \frac{a_0}{b_0} = 0.
\]

(2.25)

These conditions allow one to split the classification problem into two cases:

(a) The case \( \nu \neq -1 \) which is simpler and leads to a linearizable equation

\[
\dot{u}_0 = \frac{u_2 u_0}{u_1} + u_1 - a^2 \left( u_{-1} + \frac{u_0 u_{-2}}{u_{-1}} \right) + cu_0.
\]

(2.26)
(b) In the case when \( \nu = -1 \) we get two Theorems in terms of initial equation \( f_n \) only.

**Theorem 4** If an equation (2.1) satisfies conditions (2.21) and (2.22), then its coefficients must have the form:

\[
\begin{align*}
a_0 &= a^{(1)}(u_1, u_0)a^{(2)}(u_0, u_{-1}), \\
b_0 &= b^{(1)}(u_1, u_0)b^{(2)}(u_0, u_{-1}).
\end{align*}
\] (2.27)

**Theorem 5** If an equation (2.1) belongs to the Class I and satisfies conditions (2.21, 2.22) and (2.27), then for the functions \( a^{(2)} \) and \( b^{(1)} \) one has:

\[
\begin{align*}
\frac{\partial^2 a^{(2)}}{\partial u_{-1}^2} &= 0, \\
\frac{\partial^2 b^{(1)}}{\partial u_{1}^2} &= 0.
\end{align*}
\] (2.28)
3 List of integrable equations of Class I

We present the complete list of integrable equations of Class I (E1)-(E17) and the non-point relations between them.

The classification is usually carried out in two steps:

(a) at first one finds all integrable equations of a certain class up to point transformations $\hat{u}_0 = c_1 u_0 + c_2, \hat{t} = c_3 t, c_1 c_3 \neq 0$.

(b) then one searches for non-point transformations which relate among them the different resulting equations

Non-point transformations:

$$\hat{u}_0 = \varphi(u_k, u_{k-1}, \ldots, u_m), \ k > m.$$ (3.1)

The equation (1.1) is transformed into

$$\hat{u}_{0,t} = \hat{\Psi}(\hat{u}_2, \hat{u}_1, \hat{u}_0, \hat{u}_{-1}, \hat{u}_{-2})$$ (3.2)
Transformation (3.1) is explicit in one direction. If an equation $A$ is transformed into $B$ by a transformation (3.1), then we will write $A \rightarrow B$, so indicating the direction where it is explicit.

**Theorem 6** If an equation of the form (2.1) belongs to Class I and has a generalized symmetry (1.5), then up to point transformation it is equivalent to one of the equations (E1)-(E17).
List 1. Equations related to the Volterra equation

\[ \dot{u}_0 = u_0(u_2 - u_{-2}) \] \hspace{1cm} (E1)

\[ \dot{u}_0 = u_0^2(u_2 - u_{-2}) \] \hspace{1cm} (E2)

\[ \dot{u}_0 = (u_0^2 + u_0)(u_2 - u_{-2}) \] \hspace{1cm} (E3)

\[ \dot{u}_0 = (u_2 + u_1)(u_0 + u_{-1}) - (u_1 + u_0)(u_{-1} + u_{-2}) \] \hspace{1cm} (E4)

\[ \dot{u}_0 = (u_2 - u_1 + a)(u_0 - u_{-1} + a) + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) + b \] \hspace{1cm} (E5)

\[ \dot{u}_0 = u_2u_1u_0(u_0u_{-1} + 1) - (u_1u_0 + 1)u_0u_{-1}u_{-2} + u_0^2(u_{-1} - u_1) \] \hspace{1cm} (E6)
\begin{align*}
\hat{u}_0 &= u_1 + u_0, & \text{(T1)} \\
\hat{u}_0 &= u_1 - u_0 + a, & \text{(T2)} \\
\hat{u}_0 &= u_1 u_0, & \text{(T3)} \\
\hat{u}_0 &= u_2 u_0, & \text{(T4)} \\
\hat{u}_0 &= u_2(u_0 + 1) \quad \text{or} \quad \hat{u}_0 = (u_2 + 1)u_0. & \text{(T5)}
\end{align*}
List 2. Linearizable equations

\[ \dot{u}_0 = (T-a) \left( \frac{(u_1 + au_0 + b)(u_{-1} + au_{-2} + b)}{u_0 + au_{-1} + b} + u_0 + au_{-1} + b \right) + cu_0 + d \]  
(E7)

\[ \dot{u}_0 = \frac{u_2u_0}{u_1} + u_1 - a^2 \left( u_{-1} + \frac{u_0u_{-2}}{u_{-1}} \right) + cu_0 \]  
(E8)

In both equations \( a \neq 0 \), \( (a + 1)d = bc \).

Both equations of List 2 are related to the linear one:

\[ \dot{u}_0 = u_2 - a^2u_{-2} + cu_0/2 \]  
(3.4)

\[ \begin{array}{ccc}
\text{(3.4)} & \xrightarrow{(T3)} & \text{(E8)} \\
\text{(E7)} & \xleftarrow{(T6)} & \text{(3.5)}
\end{array} \]

\[ \hat{u}_0 = u_1 + au_0 + b. \]  
(T6)
List 3. Equations of the relativistic Toda type

\[
\dot{u}_0 = (u_0 - 1) \left( \frac{u_2(u_1 - 1)u_0}{u_1} - \frac{u_0(u_{-1} - 1)u_{-2}}{u_{-1}} - u_1 + u_{-1} \right) \quad (E9)
\]

\[
\dot{u}_0 = \frac{u_2u_1^2u_0^2(u_0u_{-1} + 1)}{u_1u_0 + 1} - \frac{(u_1u_0 + 1)u_0^2u_{-1}^2u_{-2}}{u_0u_{-1} + 1} - \frac{(u_1 - u_{-1})(2u_1u_0u_{-1} + u_1 + u_{-1})}{(u_1u_0 + 1)(u_0u_{-1} + 1)} \quad (E10)
\]

The equations of List 3 are related to the following one:

\[
\dot{u}_0 = (u_1u_0 - 1)(u_0u_{-1} - 1)(u_2 - u_{-2}) \quad (3.6)
\]

\[
\begin{align*}
(3.6) & \xrightarrow{(T^3)} (E9) & \xleftarrow{(T^7)} (E10) \\
\dot{u}_0 &= u_1u_0 + 1,
\end{align*}
\]

Eq. (3.6), which is out of Class I, it is of the relativistic Toda type.
List 4. Equations related to the INB

\[ \dot{u}_0 = u_0 (u_2 + u_1 - u_{-1} - u_{-2}) \]  
(E11)

\[ \dot{u}_0 = (u_2 - u_1 + a)(u_0 - u_{-1} + a) + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) + (u_1 - u_0 + a)(u_0 - u_{-1} + a) \]  
(E12)

\[ \dot{u}_0 = (u_2^0 + au_0)(u_2u_1 - u_{-1}u_{-2}) \]  
(E13)

\[ \dot{u}_0 = (u_1 - u_0)(u_0 - u_{-1})\left(\frac{u_2}{u_1} - \frac{u_{-2}}{u_{-1}}\right) \]  
(E14)
\[ \hat{u}_0 = u_2 u_1 (u_0 + a) \quad \text{or} \quad \hat{u}_0 = (u_2 + a) u_1 u_0, \]  
\hfill (T8)

\[ \hat{u}_0 = \frac{(u_1 - u_0)(u_0 - u_{-1})}{u_0}. \]  
\hfill (T9)
List 5. Other equations

\[ \dot{u}_0 = u_0^2(u_2 u_1 - u_{-1} u_{-2}) - u_0(u_1 - u_{-1}) \]  
(E15)

\[ \dot{u}_0 = (u_0 + 1) \left( \frac{u_2 u_0(u_1 + 1)^2}{u_1} - \frac{u_{-2} u_0(u_{-1} + 1)^2}{u_{-1}} + (1 + 2u_0)(u_1 - u_{-1}) \right) \]  
(E16)

\[ \dot{u}_0 = (u_0^2 + 1) \left( u_2 \sqrt{u_1^2 + 1} - u_{-2} \sqrt{u_{-1}^2 + 1} \right) \]  
(E17)

Equation (E15) is called the discrete Sawada-Kotera equation (S. Tsujimoto and R. Hirota).

Equation (E17) is the result of the present classification and seems to be new. There is no relation with other known equations.

\[ (E15) \xleftarrow{(T^{10})} (3.10) \xrightarrow{(T^{11})} (E16) \]  
(3.9)
\[ \dot{u}_0 = \frac{u_0 - u_1}{u_2 - u_{-1}}, \quad \text{(T10)} \]

\[ \ddot{u}_0 = -\frac{(u_1 - u_{-1})(u_0 - u_{-2})}{(u_1 - u_{-2})(u_0 - u_{-1})}, \quad \text{(T11)} \]

\[ \dot{u}_0 = \frac{(u_1 - u_0)(u_0 - u_{-1})(u_2 - u_{-2})}{(u_2 - u_{-1})(u_1 - u_{-2})}, \quad \text{(3.10)} \]

Up to linearizable transformations, both transformations (T10) and (T11) are of Miura type.
Link between the various Lists 1-5.

We exclude from the consideration (E17) as we have no information about it.

The $L - A$ pair for the equations of Volterra and relativistic Toda is given by $2 \times 2$ matrices.

The $L - A$ pair of INB and discrete Sawada-Kotera equation contained in List 4 and 5, is given by $3 \times 3$ matrices.

List 2 consists of linearizable equations.

So List 2, Lists (1,3) and Lists (4,5) are not related by non point transformations.
4 Generalized Symmetries of Key Equations

List 1. Generalized symmetries for the Volterra equation and its modifications are well-known and we just replace \( u_{n+j} \) by \( u_{n+2j} \). The simplest generalized symmetry for (E1) reads:

\[
 u_{0,\tau} = u_0(u_2(u_4 + u_2 + u_0) - u_{-2}(u_0 + u_{-2} + u_{-4})).
\]

List 2. A generalized symmetry for (E8) is obtained from the linear equation:

\[
 u_{0,\tau} = (T^4 - a^4) \left( \frac{u_{-1}u_{-3}}{u_{-2}} + \frac{u_0u_{-2}u_{-4}}{u_{-1}u_{-3}} \right). \tag{4.1}
\]

List 3. A generalized symmetry of relativistic Toda equation (E9) is:

\[
 u_{0,\tau} = (u_0 - 1)(T - T^{-1}) \left( (T + T^{-1})s_1 - (1 + T^{-1})s_2 + s_3 \right),
\]
\[s_1 = \frac{u_2 u_0 u_{-2} (u_1 - 1) (u_0 - 1) (u_{-1} - 1)}{u_1 u_{-1}}, \quad s_2 = \frac{u_1 u_{-1} (u_1 - 1) (u_0 - 1)}{u_0},\]
\[s_3 = \frac{u_1 (u_0 - 1) (u_1 u_0 - u_1 - u_0) u_{-1}^2}{u_0^2} - \frac{(u_2 u_1 - u_2 - u_1) u_0 (u_0 - 1)}{u_1}.
\]

**List 4.** Generalized symmetries INB equation has the form:

\[u_{0,\tau} = u_0 \left( T^2 + T - T^{-1} - T^{-2} \right) ((T + T^{-1}) u_1 u_{-1} + (T + 1) u_0 u_{-1} + u_0^2),\]

**List 5.** The generalized symmetry for the discrete Sawada-Kotera equation has the form:

\[u_{0,\tau} = u_0 \left( w_1 (w_3 + w_2 + w_1 + w_0) - w_{-1} (w_0 + w_{-1} + w_{-2} + w_{-3})
- u_1 (w_3 + w_{-1}) + u_{-1} (w_1 + w_{-3}) \right), \quad w_0 = u_0 (1 + u_1 u_{-1}).\]
5 Conclusion

In this talk we present the classification of the differential difference equations depending on five-point lattices of Class I. This Class is a natural subclass of the differential difference equations depending on five-point lattices from the point of view of the integrability conditions. In this Class we find 17 equations, one of which seems new. The others are related by non point transformations to key known equations.

Work to be done:

(a) extend the classification outside Class I by considering the case when (1.7) is not satisfied (Ex. (3.6)). We can call this class of equations of Class II.

(b) we can construct their simplest Bäcklund transformations which will provide integrable partial difference equations possibly on a four-point lattice not belonging to the ABS classification and its extensions.
References


