From continuous to discrete QM:
Charlier oscillator and its coalgebra symmetry
(D. Latini and D. Riglioni)

Speaker: D. Riglioni

In honour of Prof. O. Ragnisco (70th birthday)

October 20-22 2016, Burgos
Almost 10 years collaboration on Superintegrable systems

A finite dimensional \((N)\) Hamiltonian Superintegrable system is a Hamiltonian system which has a number of constants of the motion exceeding the degrees of freedom \((N)\)

\[
\{ H, X_i \} = 0 \quad ; \quad \{ X_i, X_j \} = 0 \quad ; \quad \# \{ X_i \} = N - 1
\]

\[
\{ H, I_i \} = 0 \quad ; \quad \{ I_i, X_j \} = 0 \quad ; \quad \# \{ I_i \} \leq N - 1
\]

If \(\# \{ I_i \} = N - 1\) The system is called Maximally Superintegrable

1. Exactly solvable
2. Multiseparability
3. Multiple connections with many areas in Mathematica such as special functions and group theory
Almost 10 years collaboration on Superintegrable systems

A direct approach

\[ H = p_1^2 + p_2^2 + V(x_1, x_2) \]

Establishing an integrability order for the constant of the motion \( X \)

\[ X = a(x_1, x_2)^{ij} p_i p_j + f(x_1, x_2) \]

and then solving the determining equations arising from

\[ \{ H, X \} = 0 \]

This approach generates a number of differential equations whose complexity increases enormously, as the number of degrees of freedom or the order of the symmetry, increases.
Coalgebra Approach a brief introduction

Let us consider a Poisson-Lie algebra

\[ \{ J_i, J_l \} = C_{i,l}^k J_k \] and a set of Casimir \( C_i : \{ C_i, X_l \} = 0 \)

Equipped with a coproduct \( \Delta \)

\[ \Delta : J \rightarrow J \otimes J \]

which turns out ot be an homomorphism since:

\[ \{ \Delta J_i, \Delta J_l \} = C_{i,l}^k \Delta J_k ; \{ \Delta C_i, \Delta X_l \} = 0 \]

\[ \{ \Delta D(J_i), \Delta D(J_l) \} = C_{i,l}^k \Delta D(J_k) ; \{ \Delta D(C_i), D(\Delta X_l) \} = 0 \]
The crucial point is that on the one hand the coproduct map increases the number of degrees of freedom of the generators

\[ J^3 = 1 \otimes \Delta \circ (\Delta J) = J_3 = \Delta \otimes 1 \circ (\Delta J) \]

\[ D(J^3) = 1 \otimes \Delta \circ (\Delta D(J)) = D(J_3) = \Delta \otimes 1 \circ (\Delta D(J)) \]

on the other hand the partial application of the coproduct makes possible to define elements on some subspaces

\[ D(J^2) = \Delta D(J) \otimes 1 \; ; \; D(J_2) = 1 \otimes \Delta D(J) \]

which can be used to generate new symmetries

\[ \{ D(J^3), D(C(J^3)) \} = \{ D(J^3), D(C(J^2)) \} = \{ D(J^3), D(C(J_2)) \} = 0 \]
Coalgebra symmetry Case Study: sl(2,R)

a class of 1-D Hamiltonians: \( H(x, p) = H(p^2, xp, x^2) \)

\((x, p) \in \mathbb{R}^2 \quad \{x, p\} = 1\)
Coalgebra symmetry Case Study: \( \mathfrak{sl}(2, \mathbb{R}) \)

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It can be expressed in terms of \( \mathfrak{sl}(2, \mathbb{R}) \) coalgebra generators:

\[H(p^2, xp, x^2) = H(J_+, J_-)\]

Symplectic realization of the algebra:

\[J_+ = p^2 \quad J_- = x^2 \quad J_3 = xp\]
Coalgebra symmetry Case Study: $\mathfrak{sl}(2, \mathbb{R})$

A class of 1-D Hamiltonians: $H(x, p) = H(p^2, xp, x^2)$

$$(x, p) \in \mathbb{R}^2 \quad \{x, p\} = 1$$

It can be expressed in terms of $\mathfrak{sl}(2, \mathbb{R})$ coalgebra generators:

$$H(p^2, xp, x^2) = H(J_+, J_-)$$

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$$J_+ = p^2 \quad J_- = x^2 \quad J_3 = xp$$

Poisson-Lie $\mathfrak{sl}(2, \mathbb{R})$ coalgebra:

$$\{J_-, J_+\} = 4J_3 \quad \{J_3, J_\pm\} = \pm 2J_\pm$$

Casimir function: $C(J_\pm, J_3) \doteq J_+J_- - J_3^2 = 0$ (in the given representation)
The (Poisson) algebra \( \mathfrak{sl}(2, \mathbb{R}) \) is then equipped with a map called \textit{coproduct}:

\[
\Delta : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})
\]
The (Poisson) algebra $\mathfrak{sl}(2, \mathbb{R})$ is then equipped with a map called \textit{coproduct}:

$$\Delta(J_\sigma) \doteq J_\sigma \otimes 1 + 1 \otimes J_\sigma, \quad \Delta(1) \doteq 1 \otimes 1 \quad (\text{with } \sigma = \pm, 3)$$
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Properties:

1. **Homomorphism property:**

   \[
   \{\Delta(J_-), \Delta(J_+)\} = 4\Delta(J_3) \quad \{\Delta(J_3), \Delta(J_\pm)\} = \pm 2\Delta(J_\pm) \n   \]

2. **Coassociativity:**

   \[
   (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \n   \]
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2. **Coassociativity:**

$$\begin{array}{ccc}
\mathfrak{sl}(2, \mathbb{R}) & \xrightarrow{\Delta} & \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R}) \\
\downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\
\mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R}) & \xrightarrow{\text{id} \otimes \Delta} & \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})
\end{array}$$
The (Poisson) algebra $\mathfrak{sl}(2, \mathbb{R})$ is then equipped with a map called \textit{coproduct}: 

$$\Delta(J_{\sigma}) \doteq J_{\sigma} \otimes 1 + 1 \otimes J_{\sigma}, \quad \Delta(1) \doteq 1 \otimes 1 \quad \text{(with } \sigma = \pm, 3)$$

Properties:

1. \textit{Homomorphism property:}

$$\{\Delta(J_{-}), \Delta(J_{+})\} = 4\Delta(J_{3}) \quad \{\Delta(J_{3}), \Delta(J_{\pm})\} = \pm 2\Delta(J_{\pm})$$

2. \textit{Coassociativity:}

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$$

The new copies generated by the coproduct map close the $\mathfrak{sl}(2, \mathbb{R})$ algebra

[physically $\longrightarrow$ \textit{multidimensional extension}]

2. An object on $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})$ can be defined in two ways

[physically $\longrightarrow$ \textit{superintegrability properties}]

O. Ragnisco (Roma Tre University)
Central potentials and planar motion as coalgebra symmetry

Let us consider the following 1-D Hamiltonian:

\[ H = F(x^2)p^2 + V(x^2) \]

In terms of coalgebra generators it becomes:

\[ H = F(J_-)J_+ + V(J_-) \]

By applying twice the coproduct we obtain:

\[ (1 \otimes \Delta) \circ \Delta J \rightarrow \]

\[ D(J^3_+) = p_1^2 + p_2^2 + p_3^2 = |p|^2 \]

\[ D(J^3_3) = x_1p_1 + x_2p_2 + x_3p_3 = x \cdot p \rightarrow H = F(|x|^2)|p|^2 + V(|x|^2) \]

\[ D(J^3_-) = x_1^2 + x_2^2 + x_3^2 = |x|^2 \]
Central potentials and planar motion as coalgebra symmetry

Because of the coalgebra symmetry the Hamiltonian $H$ Poisson commutes with

$C^3 = J^3_+ J^3_- - J^3_3 \rightarrow$

$D(C^3) = (x_1 p_2 - x_2 p_1)^2 + (x_1 p_3 - x_3 p_1)^2 + (x_3 p_2 - x_2 p_3)^2 = L^2_1 + L^2_2 + L^2_3$

$D(J^2_+) = p^2_1 + p^2_2$

$D(J^2_3) = x_1 p_1 + x_2 p_2 \rightarrow C^2 = J^2_+ J^2_- - J^2_3 \rightarrow D(C^2) = (x_1 p_2 - x_2 p_1)^2 = L^2_3$

$D(J^2_-) = x^2_2 + x^2_3$

$D(J^2_{+,2}) = p^2_2 + p^2_3$

$D(J^2_{3,2}) = x_2 p_2 + x_3 p_3 \rightarrow C_2 = J^2_{+,2} J^2_{-,2} - J^2_{3,2} \rightarrow D(C_2) = (x_2 p_3 - x_3 p_2)^2 = L^2_1$

$D(J^2_{-,2}) = x^2_2 + x^2_3$
Canonical transformations generating new Coalgebraic superintegrable systems

Let us consider the 2-D polar representation of the coalgebra generators

\[ D(J_+) = p_r^2 + \frac{p_\theta^2}{r^2} \; ; \; D(J_-) = r^2 \; ; \; D(J_3) = rp_r \]

any central Hamiltonian can be regarded as

\[ H = p_r^2 + \frac{p_\theta^2}{r^2} + V(r^2) \rightarrow H = \frac{J_3^2}{J_-} + \frac{C}{J_-} + V(J_-) \] it is possible to induce new superintegrable systems by applying a canonical transformation e.g. :

\[
\begin{align*}
\theta &= \beta \theta' \\
p_\theta &= \frac{p_\theta'}{\beta^2}
\end{align*}
\]

\[ H = \frac{J_3^2}{J_-} + \frac{C}{J_-} + V(J_-) \rightarrow H' = \frac{J_3^2}{J_-} + \frac{C}{\beta^2 J_-} + V(J_-) \]

if \( \beta \in \mathbb{Q} \) then \( H' \) Higher order superintegrable system
"Gauge" transformations generating new Coalgebraic superintegrable systems with spin interactions

Let us consider a 2-D radial quantum realization of the $\mathfrak{sl}(2)$ coalgebra

\[ D(J_+^2) = -(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) ; \quad D(J_-^2) = r^2 ; \quad D(J_3) = -i(r \partial_r + 1) ; \quad \{,\} \rightarrow [,] \]

\[ H = -\frac{1}{2}(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) + V(r) \rightarrow H = \frac{1}{2} J_+ + V(J_-) \]

Let us apply a similarity transformation

\[ e^{-i\gamma\theta} He^{i\gamma\theta} = H + \frac{\gamma}{r^2}(-i\partial_\theta) + \frac{\gamma^2}{r^2} \rightarrow H = \frac{1}{2} J_+ + \gamma \frac{\sqrt{C^2 + 1}}{J_-} + \frac{\gamma^2}{J_-} + V(J_-) \]

this new coalgebraic system can now be embedded in a 3-D space by the $\Delta$ generating a spin interaction

\[ \sqrt{C^3 + 1} = \sqrt{L_1^2 + L_2^2 + L_3^2 + \frac{1}{4}} = \sigma_i L_i + \frac{1}{2} \]
Coalgebra and discrete quantum mechanics

1. Establishing a link between ordinary QM and discrete QM†

2. Generalizing discrete quantum mechanical systems to the ND-case and analyze their superintegrability properties by means of the coalgebra‡

3. Solving the (discrete) spectral problems on the lattice


Coalgebra symmetry and the harmonic oscillator

The harmonic oscillator as coalgebra $\mathfrak{sl}(2, \mathbb{R})$ element: the classical case

Harmonic oscillator Hamiltonian: $H(x, p) = \frac{p^2 + x^2}{2}$

$(x, p) \in \mathbb{R}^2 \quad \{x, p\} = 1$

It can be expressed in terms of $\mathfrak{sl}(2, \mathbb{R})$ coalgebra generators:

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Symplectic realization of the algebra:

$$J_+ = p^2 \quad J_- = x^2 \quad J_3 = xp$$

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$(\hat{x}, \hat{p}) \in \mathcal{H} \quad [\hat{x}, \hat{p}] = i\hbar$

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It can be expressed in terms of \( \mathfrak{sl}(2, \mathbb{R}) \) coalgebra generators:

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$(\hat{x}, \hat{p}) \in \mathcal{H}$ \hspace{1cm} $[\hat{x}, \hat{p}] = i\hbar$ \hspace{1cm} $(\hat{x} \equiv x, \hat{p} \equiv -i\hbar \partial_x)$

It can be expressed in terms of $\mathfrak{sl}(2, \mathbb{R})$ coalgebra generators:

$\hat{H}(\hat{J}_+, \hat{J}_-) = \frac{\hat{J}_+ + \hat{J}_-}{2}$

Differential realization of the algebra:

$\hat{J}_+ = -\hbar^2 \partial_{xx}$ \hspace{1cm} $\hat{J}_- = x^2$ \hspace{1cm} $\hat{J}_3 = -i\hbar(x\partial_x + 1/2)$

Poisson-Lie $\mathfrak{sl}(2, \mathbb{R})$ coalgebra:

$\{J_-, J_+\} = 4J_3 \hspace{1cm} \{J_3, J_\pm\} = \pm 2J_\pm$

Casimir function: $C(J_\pm, J_3) \doteq J_+ J_- - J_3^2 = 0$
Coalgebra symmetry and the harmonic oscillator
The harmonic oscillator as coalgebra $\mathfrak{sl}(2, \mathbb{R})$ element: the quantum case

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Quantum-Lie $\mathfrak{sl}(2, \mathbb{R})$ coalgebra:

$[\hat{J}_-, \hat{J}_+] = 4i\hbar \hat{J}_3$ \hspace{1cm} $[\hat{J}_3, \hat{J}_\pm] = \pm 2i\hbar \hat{J}_\pm$

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**Quantum-Lie \( \mathfrak{sl}(2, \mathbb{R}) \) coalgebra:**

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[\hat{J}_-, \hat{J}_+] = 4i\hbar \hat{J}_3 \quad [\hat{J}_3, \hat{J}_\pm] = \pm 2i\hbar \hat{J}_\pm
\]

**Casimir operator:**

\[
\hat{C}(\hat{J}_\pm, \hat{J}_3) \doteq 1/2 [\hat{J}_+, \hat{J}_-]_+ - \hat{J}_3^2 = (i\hbar)^2 1/2 (1/2 + 1)
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**Casimir operator:**

$$\Delta(\hat{C}(\hat{J}_\pm, \hat{J}_3)) = (-i\hbar x_1 \partial x_2 + i\hbar x_2 \partial x_1)^2 - \hbar^2$$
Coalgebra symmetry and the harmonic oscillator

The harmonic oscillator as coalgebra $\mathfrak{sl}(2, \mathbb{R})$ element: the one-dimensional spectral problem

One-dimensional spectral problem:

Factorization: \[ \hat{H}(\hat{J}_-, \hat{J}_+) = \hat{a}^\dagger (\hat{J}_-, \hat{J}_3) \hat{a} (\hat{J}_-, \hat{J}_3) + \frac{\hbar}{2} \]
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Ladder operators:

\[
\begin{align*}
\hat{a}(\hat{J}_-, \hat{J}_3) &\doteq \frac{i}{\sqrt{2}} \hat{J}_-^{-\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \frac{\hbar}{\sqrt{2}} \partial_x + \frac{x}{\sqrt{2}} \\
\hat{a}^\dagger(\hat{J}_-, \hat{J}_3) &\doteq -\frac{i}{\sqrt{2}} \hat{J}_-^{-\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = -\frac{\hbar}{\sqrt{2}} \partial_x + \frac{x}{\sqrt{2}}
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\end{align*}
\]

Vacuum state: \[ \hat{a}(\hat{J}_-, \hat{J}_3)\Psi_0(x) = 0 \implies \Psi_0(x) \propto \exp\left(-\frac{x^2}{2\hbar}\right) \]
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Ladder operators:
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\begin{align*}
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\hat{a}^\dagger(\hat{J}_-, \hat{J}_3) &= -\frac{i}{\sqrt{2}} \hat{J}^{-\frac{1}{2}} \left[ \hat{J}_3 + \frac{i \hbar}{2} \right] + \frac{1}{\sqrt{2}} \hat{J}_{--} = -\frac{\hbar}{\sqrt{2}} \partial_x + \frac{x}{\sqrt{2}}
\end{align*}
\]

Vacuum state: \[ \hat{a}(\hat{J}_-, \hat{J}_3)\Psi_0(x) = 0 \quad \longrightarrow \quad \Psi_0(x) \propto \exp\left(-\frac{x^2}{2\hbar}\right) \]

Eigenfunctions: \[ \Psi_n(x) \propto [\hat{a}^\dagger(\hat{J}_-, \hat{J}_3)]^n\Psi_0(x) = \left(\frac{\hbar}{2}\right)^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{\hbar}}\right)\Psi_0(x) \]
Coalgebra symmetry and the harmonic oscillator

The harmonic oscillator as coalgebra \( \mathfrak{sl}(2, \mathbb{R}) \) element: the one-dimensional spectral problem

One-dimensional spectral problem:

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Eigenfunctions:
\[ \Psi_n(x) \propto [\hat{a}^\dagger(\hat{J}_- \hat{J}_3)]^n \Psi_0(x) = \left( \frac{\hbar}{2} \right)^\frac{n}{2} H_n \left( \frac{x}{\sqrt{\hbar}} \right) \Psi_0(x) \]

Spectrum:
\[ \hat{H}(\hat{J}_- \hat{J}_+)\Psi_n(x) = \hbar(n + 1/2)\Psi_n(x) \quad (n = 0, 1, \ldots) \]
Coalgebra symmetry and the harmonic oscillator

The harmonic oscillator as coalgebra $sl(2, \mathbb{R})$ element: the two-dimensional spectral problem

$\Delta$(One-dimensional spectral problem):

**Factorization:**

\[
\Delta(\hat{H}(\hat{J}_-, \hat{J}_+)) = \sum_{k=1}^{2} \hat{a}_k^\dagger(\hat{J}(-,k), \hat{J}(3,k)) \hat{a}_k(\hat{J}(-,k), \hat{J}(3,k)) + \hbar,
\]

**Ladder operators:**

\[
\begin{align*}
\hat{a}_k(\hat{J}(-,k), \hat{J}(-,k)) &\doteq \frac{i}{\sqrt{2}} \hat{J}(-,k) \left( \hat{J}(-,k) + \frac{i\hbar}{2} \right) + \frac{1}{\sqrt{2}} \hat{J}^\frac{1}{2}(-,k) \\
\hat{a}_k^\dagger(\hat{J}(-,k), \hat{J}(-,k)) &\doteq -\frac{i}{\sqrt{2}} \hat{J}(-,k) \left( \hat{J}(-,k) + \frac{i\hbar}{2} \right) + \frac{1}{\sqrt{2}} \hat{J}^\frac{1}{2}(-,k)
\end{align*}
\]

**Vacuum state:**

\[
\hat{a}_k(\hat{J}(-,k), \hat{J}(3,k)) \Psi_0(x_k) = 0 \quad \rightarrow \quad \Psi_{(0,0)}(x_1, x_2) \propto \exp\left(-\frac{x_1^2 + x_2^2}{2\hbar}\right)
\]

**Eigenfunctions:**

\[
\Psi_{(n,m)}(x_1, x_2) \propto \left(\frac{\hbar}{2}\right)^{\frac{n+m}{2}} \frac{n+m}{\sqrt{2}} H_n \left(\frac{x_1}{\sqrt{\hbar}}\right) H_m \left(\frac{x_2}{\sqrt{\hbar}}\right) \Psi_{(0,0)}(x_1, x_2)
\]

**Spectrum:**

\[
\hat{H}^{(2)} \Psi_{(n,m)}(x_1, x_2) = \hbar(n + m + 1) \Psi_{(n,m)}(x_1, x_2) \quad (n, m = 0, 1, \ldots)
\]
Discrete QM and the harmonic oscillator

Linking oQM and dQM: canonical transformations

Let us consider the following change of variables ($\lambda > 0$, $-1/2\lambda \leq \bar{x} < \infty$)

\[
\begin{align*}
    x(\bar{x}, \bar{p}) &= \frac{\sqrt{1 + 2\lambda\bar{x} \cos(\lambda\bar{p})} - 1}{\lambda} \\
    p(\bar{x}, \bar{p}) &= \frac{\sqrt{1 + 2\lambda\bar{x} \sin(\lambda\bar{p})}}{\lambda}
\end{align*}
\]

\[\{x(\bar{x}, \bar{p}), p(\bar{x}, \bar{p})\}(\bar{x}, \bar{p}) = 1\]
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\]

Hamiltonian:

\[
H(\bar{x}, \bar{p}) = \frac{\bar{J}_+ + \bar{J}_-}{2} = \frac{1 + \lambda \bar{x} - \sqrt{1 + 2\lambda \bar{x}} \cos(\lambda \bar{p})}{\lambda^2}
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\begin{align*}
\begin{cases}
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p(\bar{x}, \bar{p}) = \frac{\sqrt{1 + 2\lambda \bar{x} \sin(\lambda \bar{p})}}{\lambda}
\end{cases}
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\]

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\]

Generators:

\[
\begin{align*}
\bar{J}_+ &= p^2(\bar{x}, \bar{p}) = \frac{(1 + 2\lambda \bar{x}) \sin^2(\lambda \bar{p})}{\lambda^2} \\
\bar{J}_- &= x^2(\bar{x}, \bar{p}) = \frac{(1 + 2\lambda \bar{x}) \cos^2(\lambda \bar{p}) - 2\sqrt{1 + 2\lambda \bar{x} \cos(\lambda \bar{p})} + 1}{\lambda^2} \\
\bar{J}_3 &= x(\bar{x}, \bar{p})p(\bar{x}, \bar{p}) = \frac{(1 + 2\lambda \bar{x}) \cos(\lambda \bar{p}) - \sqrt{1 + 2\lambda \bar{x}}}{\lambda^2} \sin(\lambda \bar{p})
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\begin{align*}
\bar{J}_+ &= p^2(\bar{x}, \bar{p}) = \frac{(1 + 2\lambda\bar{x}) \sin^2(\lambda\bar{p})}{\lambda^2} \\
\bar{J}_- &= x^2(\bar{x}, \bar{p}) = \frac{(1 + 2\lambda\bar{x}) \cos^2(\lambda\bar{p}) - 2\sqrt{1 + 2\lambda\bar{x}} \cos(\lambda\bar{p}) + 1}{\lambda^2} \\
\bar{J}_3 &= x(\bar{x}, \bar{p})p(\bar{x}, \bar{p}) = \frac{(1 + 2\lambda\bar{x}) \cos(\lambda\bar{p}) - \sqrt{1 + 2\lambda\bar{x}} \sin(\lambda\bar{p})}{\lambda^2}
\end{align*}
\]

\(\mathfrak{sl}(2, \mathbb{R})\) algebra:

\[
\{\bar{J}_-, \bar{J}_+\}_{(\bar{x}, \bar{p})} = 4\bar{J}_3 \quad \{\bar{J}_3, \bar{J}_\pm\}_{(\bar{x}, \bar{p})} = \pm 2\bar{J}_\pm
\]
Let us perform the canonical quantization: \( \bar{p} \to \hat{p} = -i\hbar \partial_x \), \( \bar{x} \to \hat{x} = \bar{x} \).
Discrete QM and the harmonic oscillator

Linking oQM and dQM: quantization

Let us perform the canonical quantization: \( \bar{p} \rightarrow \hat{p} = -i\hbar \partial_{\bar{x}} \), \( \bar{x} \rightarrow \hat{x} = \bar{x} \):

\[
\begin{align*}
\hat{x}(\bar{x}, -i\hbar \partial_{\bar{x}}) &= \frac{\sqrt{1 + 2\lambda(\bar{x} + \lambda\hbar)} e^{\lambda\hbar \partial_{\bar{x}}} + \sqrt{1 + 2\lambda \bar{x}} e^{-\lambda\hbar \partial_{\bar{x}}} - 2}{2\lambda} \\
\hat{p}(\bar{x}, -i\hbar \partial_{\bar{x}}) &= \frac{\sqrt{1 + 2\lambda(\bar{x} + \lambda\hbar)} e^{\lambda\hbar \partial_{\bar{x}}} - \sqrt{1 + 2\lambda \bar{x}} e^{-\lambda\hbar \partial_{\bar{x}}}}{2i\lambda}
\end{align*}
\]

\([\hat{x}(\bar{x}, -i\hbar \partial_{\bar{x}}), \hat{p}(\bar{x}, -i\hbar \partial_{\bar{x}})] = i\hbar\)
Let us perform the canonical quantization: \( \bar{p} \rightarrow \hat{p} = -i\hbar \partial_{\bar{x}} \), \( \bar{x} \rightarrow \hat{x} = \bar{x} \):

\[
\begin{align*}
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\end{align*}
\]

\[ [\hat{x}(\bar{x}, -i\hbar \partial_{\bar{x}}), \hat{p}(\bar{x}, -i\hbar \partial_{\bar{x}})] = i\hbar \]

\[ \hat{H} = \frac{\hat{J}_+ + \hat{J}_-}{2} = \frac{(\lambda^2 \hbar + 2\lambda \bar{x} + 2) - \sqrt{2\lambda(\bar{x} + \lambda \hbar)} + 1 \hat{T}^+ - \sqrt{2\lambda \bar{x} + 1} \hat{T}^-}{2\lambda^2} \]
Discrete QM and the harmonic oscillator

Linking oQM and dQM: quantization

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\end{align*}
\]

\[
[\hat{x}(\bar{x}, -i\hbar \partial_{\bar{x}}), \hat{p}(\bar{x}, -i\hbar \partial_{\bar{x}})] = i\hbar
\]

\[
\hat{H} = \frac{\hat{J}_+ + \hat{J}_-}{2} = \frac{(\lambda^2\hbar + 2\lambda\bar{x} + 2) - \sqrt{2\lambda(\bar{x} + \lambda\hbar)} + 1}{2\lambda^2} \hat{T}^+ - \sqrt{2\lambda\bar{x} + 1} \hat{T}^-
\]

the operators \( \hat{T}^\pm \equiv e^{\pm\lambda\hbar \partial_{\bar{x}}} \) are such that:

\[
\hat{T}^\pm f(\bar{x}) = f(\bar{x} \pm \hbar\lambda) \quad \text{shift operators!}
\]
Discrete QM and the harmonic oscillator
Linking oQM and dQM: quantization

Discrete representation of the $\mathfrak{sl}(2, \mathbb{R})$:

\[
\hat{J}_+ = \frac{2(\lambda^2 \hbar + 2\lambda \bar{x} + 1) - \sqrt{2\lambda}(\lambda \hbar + \bar{x}) + 1 \sqrt{2\lambda(2\lambda \hbar + \bar{x}) + 1} \hat{T}^{++} - \sqrt{2\lambda \bar{x} + 1} \sqrt{2\lambda(\bar{x} - \lambda \hbar) + 1} \hat{T}^{--}}{4\lambda^2},
\]

\[
\hat{J}_- = \frac{2(\lambda^2 \hbar + 2\lambda \bar{x} + 3) + \sqrt{2\lambda} \bar{x} + 1 \left( \sqrt{2\lambda(\bar{x} - \lambda \hbar) + 1} \hat{T}^{--} - 4\hat{T}^{--} \right) + \sqrt{2\lambda(\lambda \hbar + \bar{x}) + 1} \left( \sqrt{2\lambda(2\lambda \hbar + \bar{x}) + 1} \hat{T}^{++} - 4\hat{T}^{++} \right)}{4\lambda^2},
\]

\[
\hat{J}_3 = \frac{i}{4\lambda} \left( \sqrt{2\lambda(\bar{x} - \lambda \hbar) + 1} \hat{T}^{--} - 2\hat{T}^{--} \right) + \sqrt{2\lambda(\lambda \hbar + \bar{x}) + 1} \left( 2\hat{T}^{++} - \sqrt{2\lambda(2\lambda \hbar + \bar{x}) + 1} \hat{T}^{++} \right)
\]

where we defined: \( \hat{T}^{++} = e^{2\lambda \hbar \partial \bar{x}} \) \( \hat{T}^{--} = e^{-2\lambda \hbar \partial \bar{x}} \)
Discrete QM and the harmonic oscillator

Linking oQM and dQM: quantization

Discrete representation of the $\mathfrak{sl}(2, \mathbb{R})$:

\begin{align*}
\hat{J}_+ &= \frac{2(\lambda^2 h + 2\lambda \bar{x} + 1) - \sqrt{2\lambda(h + \bar{x}) + 1} \sqrt{2\lambda(2\lambda h + \bar{x}) + 1} \hat{T}^{++} - \sqrt{2\lambda \bar{x} + 1} \sqrt{2\lambda \bar{x} - \lambda h + 1} \hat{T}^{--}}{4\lambda^2} \\
\hat{J}_- &= \frac{2(\lambda^2 h + 2\lambda \bar{x} + 3) + \sqrt{2\lambda \bar{x} + 1} \left( \sqrt{2\lambda(\bar{x} - \lambda h + 1) \hat{T}^{--} - 4\hat{T}^-} \right) + \sqrt{2\lambda(h + \bar{x}) + 1} \left( \sqrt{2\lambda(2\lambda h + \bar{x}) + 1} \hat{T}^{++} - 4\hat{T}^+ \right)}{4\lambda^2} \\
\hat{J}_3 &= i \frac{\sqrt{2\lambda \bar{x} + 1} \left( \sqrt{2\lambda(\bar{x} - \lambda h + 1) \hat{T}^{--} - 2\hat{T}^-} \right) + \sqrt{2\lambda(h + \bar{x}) + 1} \left( 2\hat{T}^+ - \sqrt{2\lambda(2\lambda h + \bar{x}) + 1} \hat{T}^{++} \right)}{4\lambda^2}
\end{align*}

where we defined:

\[ \hat{T}^{++} \doteq e^{2\lambda h \partial \bar{x}} \quad \hat{T}^{--} \doteq e^{-2\lambda h \partial \bar{x}} \]

$\mathfrak{sl}(2, \mathbb{R})$ algebra:

\[ [\hat{J}_-, \hat{J}_+] = 4i\hbar \hat{J}_3 \quad [\hat{J}_3, \hat{J}_\pm] = \pm 2i\hbar \hat{J}_\pm \]

Casimir operator:

\[ \hat{C}(\hat{J}_\pm, \hat{J}_3) \doteq \frac{1}{2} [\hat{J}_+, \hat{J}_-]_+ - \hat{J}_3^2 = (i\hbar)^2 \frac{1}{2} \left( \frac{1}{2} + 1 \right) \]
Discrete QM and the harmonic oscillator

Linking oQM and dQM: quantization

Discrete representation of the $\mathfrak{sl}(2, \mathbb{R})$:

\[ \hat{J}_+ = \frac{2(\lambda^2 \hbar + 2\lambda \bar{x} + 1) - \sqrt{2\lambda(\lambda \hbar + \bar{x}) + 1} \sqrt{2\lambda(2\lambda \hbar + \bar{x}) + 1} \hat{T}^+ + \frac{\sqrt{2\lambda \bar{x} + 1} \sqrt{2\lambda(\bar{x} - \lambda \hbar) + 1}}{4\lambda^2} \hat{T}^-}{4\lambda^2} \]

\[ \hat{J}_- = \frac{2(\lambda^2 \hbar + 2\lambda \bar{x} + 3) + \sqrt{2\lambda \bar{x} + 1} \left( \sqrt{2\lambda(\bar{x} - \lambda \hbar) + 1} \hat{T}^- + 4 \hat{T}^- \right) + \sqrt{2\lambda(\lambda \hbar + \bar{x}) + 1} \left( \sqrt{2\lambda(2\lambda \hbar + \bar{x}) + 1} \hat{T}^+ + 4 \hat{T}^+ \right)}{4\lambda^2} \]

\[ \hat{J}_3 = i \frac{\sqrt{2\lambda \bar{x} + 1} \left( \sqrt{2\lambda(\bar{x} - \lambda \hbar) + 1} \hat{T}^- - 2 \hat{T}^- \right) + \sqrt{2\lambda(\lambda \hbar + \bar{x}) + 1} \left( 2 \hat{T}^+ - \sqrt{2\lambda(2\lambda \hbar + \bar{x}) + 1} \hat{T}^+ \right)}{4\lambda^2} \]

where we defined: \[ \hat{T}^+ = e^{2\lambda \hbar \partial x} \quad \hat{T}^- = e^{-2\lambda \hbar \partial x} \]

$\mathfrak{sl}(2, \mathbb{R})$ algebra:

\[ [\hat{J}_-, \hat{J}_+] = 4i \hbar \hat{J}_3 \quad [\hat{J}_3, \hat{J}_\pm] = \pm 2i \hbar \hat{J}_\pm \]

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\[ \hat{C}(\hat{J}_\pm, \hat{J}_3) = \frac{1}{2} [\hat{J}_-, \hat{J}_+] + \hat{J}_3^2 = (i \hbar)^2 \frac{1}{2} (\frac{1}{2} + 1) \]

In the continuous $\lambda \to 0$ limit the difference realization of the $\mathfrak{sl}(2, \mathbb{R})$ algebra collapses to the standard differential realization.
Discrete QM and the harmonic oscillator
Linking oQM and dQM: the real discrete quantum mechanics

The crucial point is that by defining the quantities:

\[ B(\bar{x}, \lambda) \equiv \frac{1}{2\lambda^2} > 0 \quad D(\bar{x}, \lambda) \equiv \frac{1 + 2\lambda\bar{x}}{2\lambda^2} \geq 0 \]
Discrete QM and the harmonic oscillator

Linking oQM and dQM: the real discrete quantum mechanics

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\[ B(\bar{x}, \lambda) \doteq \frac{1}{2\lambda^2} > 0 \quad \text{and} \quad D(\bar{x}, \lambda) \doteq \frac{1 + 2\lambda\bar{x}}{2\lambda^2} \geq 0 \]

the Hamiltonian can be cast in the following form:

\[
\hat{H} = -\sqrt{B(\bar{x}, \lambda)} \hat{T}^+ \sqrt{D(\bar{x}, \lambda)} - \sqrt{D(\bar{x}, \lambda)} \hat{T}^- \sqrt{B(\bar{x}, \lambda)} + (B(\bar{x}, \lambda) + D(\bar{x}, \lambda) + \hbar/2)
\]
Discrete QM and the harmonic oscillator
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The crucial point is that by defining the quantities:

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This Hamiltonian belongs to the realm of (real) discrete quantum mechanics!†

\[ (\lambda \to 0 \implies \text{continuous limit}) \]

Discrete QM and the harmonic oscillator

Linking oQM and dQM: the one-dimensional spectral problem

One-dimensional (discrete) spectral problem:

Factorization: \[ \hat{H}(\hat{J}_-, \hat{J}_+) = \frac{\hat{J}_+ + \hat{J}_-}{2} = \hat{a}^\dagger(\hat{J}_-, \hat{J}_3) \hat{a}(\hat{J}_-, \hat{J}_3) + \frac{\hbar}{2} \]
Discrete QM and the harmonic oscillator
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Ladder Op.: \[
\begin{align*}
\hat{a}(\hat{J}_-, \hat{J}_3) &\doteq \frac{i}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \hat{T}^+ \sqrt{D(x, \lambda)} - \sqrt{B(x, \lambda)} \\
\hat{a}^\dagger(\hat{J}_-, \hat{J}_3) &\doteq -\frac{i}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \sqrt{D(x, \lambda)} \hat{T}^- - \sqrt{B(x, \lambda)}
\end{align*}
\]
Discrete QM and the harmonic oscillator

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\begin{align*}
\hat{a}(\hat{J}_-, \hat{J}_3) &= \frac{i}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \hat{T}^+ \sqrt{D(\bar{x}, \lambda)} - \sqrt{B(\bar{x}, \lambda)} \\
\hat{a}^\dagger(\hat{J}_-, \hat{J}_3) &= -\frac{i}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \sqrt{D(\bar{x}, \lambda)} \hat{T}^- - \sqrt{B(\bar{x}, \lambda)}
\end{align*}
\]

Vacuum state: \( \hat{a}(\hat{J}_-, \hat{J}_3)\Phi_0(\bar{x}) = 0 \quad \longrightarrow \quad \Phi_0(\bar{x}) \propto \sqrt{\frac{\left(\frac{1}{2\hbar \lambda^2}\right)\left(\frac{\bar{x}}{\hbar \lambda} + \frac{1}{2\hbar \lambda^2}\right)}{\left(\frac{\bar{x}}{\hbar \lambda} + \frac{1}{2\hbar \lambda^2}\right)!}} \)
Discrete QM and the harmonic oscillator

Linking oQM and dQM: the one-dimensional spectral problem

One-dimensional (discrete) spectral problem:

Factorization: \[ \hat{H}(\hat{J}_-, \hat{J}_+) = \frac{\hat{J}_+ + \hat{J}_-}{2} = \hat{a}^\dagger(\hat{J}_-, \hat{J}_3)\hat{a}(\hat{J}_-, \hat{J}_3) + \frac{\hbar}{2} \]

Ladder Op.: \[
\begin{align*}
\hat{a}(\hat{J}_-, \hat{J}_3) &\doteq \frac{i}{\sqrt{2}} \hat{J}_-^{-\frac{1}{2}} [\hat{J}_3 + i\hbar] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \hat{T}^+ \sqrt{D(\bar{x}, \lambda)} - \sqrt{B(\bar{x}, \lambda)} \\
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\end{align*}
\]

Vacuum state: \[ \hat{a}(\hat{J}_-, \hat{J}_3)\Phi_0(\bar{x}) = 0 \quad \rightarrow \quad \Phi_0(\bar{x}) \propto \sqrt{\left(\frac{1}{2\hbar^2}\right)\left(\frac{\bar{x}^2}{\chi_h} + \frac{1}{2\hbar^2}\right)} \left(\frac{\bar{x}^2}{\chi_h} + \frac{1}{2\hbar^2}\right)!
\]

Eigenf.: \[ \Phi_n(\bar{x}) \propto [\hat{a}^\dagger(\hat{J}_-, \hat{J}_3)]^n \Phi_0(\bar{x}) = \left(\frac{-1}{\sqrt{2}\lambda}\right)^n C_n \left(\frac{\bar{x}}{\chi_h} + \frac{1}{2\hbar^2}, \frac{1}{2\hbar^2}\right) \Phi_0(\bar{x}) \]
Discrete QM and the harmonic oscillator

Linking oQM and dQM: the one-dimensional spectral problem

One-dimensional (discrete) spectral problem:

Factorization: \( \hat{H}(\hat{J}_-, \hat{J}_+) = \frac{\hat{J}_+ + \hat{J}_-}{2} = \hat{a}^\dagger(\hat{J}_-, \hat{J}_3) \hat{a}(\hat{J}_-, \hat{J}_3) + \frac{\hbar}{2} \)

Ladder Op.: \[
\begin{align*}
\hat{a}(\hat{J}_-, \hat{J}_3) &\doteq \frac{i}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \hat{T}^+ \sqrt{D(\bar{x}, \lambda)} - \sqrt{B(\bar{x}, \lambda)} \\
\hat{a}^\dagger(\hat{J}_-, \hat{J}_3) &\doteq -\frac{i}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} [\hat{J}_3 + \frac{i\hbar}{2}] + \frac{1}{\sqrt{2}} \hat{J}_-^{\frac{1}{2}} = \sqrt{D(\bar{x}, \lambda)} \hat{T}^- - \sqrt{B(\bar{x}, \lambda)}
\end{align*}
\]

Vacuum state: \( \hat{a}(\hat{J}_-, \hat{J}_3) \Phi_0(\bar{x}) = 0 \quad \rightarrow \quad \Phi_0(\bar{x}) \propto \sqrt{\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2h\lambda^2}}\right)\left(\frac{\bar{x}}{\frac{1}{2h\lambda^2}} + \frac{1}{\frac{1}{2h\lambda^2}}\right)!}
\)

Eigenf.: \( \Phi_n(\bar{x}) \propto [\hat{a}^\dagger(\hat{J}_-, \hat{J}_3)]^n \Phi_0(\bar{x}) = \left(\frac{-1}{\sqrt{2}\lambda}\right)^n C_n \left(\frac{\bar{x}}{\frac{1}{2h\lambda^2}} + \frac{1}{\frac{2h\lambda^2}{2h\lambda^2}}, \frac{1}{\frac{2h\lambda^2}{2h\lambda^2}}\right) \Phi_0(\bar{x}) \)

Spectrum: \( \hat{H}(\hat{J}_-, \hat{J}_+) \Phi_n(\bar{x}) = \hbar(n + 1/2) \Phi_n(\bar{x}) \quad (n = 0, 1, \ldots) \)
Discrete QM and the harmonic oscillator

Linking oQM and dQM: the two-dimensional spectral problem

\[ \Delta(\text{One-dimensional (discrete) spectral problem)}): \]

Factorization: \[ \Delta(\hat{H}(\hat{J}_-, \hat{J}_+)) = \sum_{k=1}^{2} \hat{a}_k^\dagger(\hat{J}_{-,k}, \hat{J}_{3,k})\hat{a}_k(\hat{J}_{-,k}, \hat{J}_{3,k}) + \hbar \]

LO: \[
\begin{align*}
\hat{a}_k(\hat{J}_{-,k}, \hat{J}_{-,k}) &= \frac{i}{\sqrt{2}} \hat{J}_{-,k}^{-\frac{1}{2}}(\hat{J}_{3,k} + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}} \hat{J}_{-,k}^{\frac{1}{2}} = \hat{T}_k^+ \sqrt{D(\bar{x}_k, \lambda)} - \sqrt{B(\bar{x}_k, \lambda)} \\
\hat{a}_k^\dagger(\hat{J}_{-,k}, \hat{J}_{-,k}) &= -\frac{i}{\sqrt{2}} \hat{J}_{-,k}^{-\frac{1}{2}}(\hat{J}_{3,k} + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}} \hat{J}_{-,k}^{\frac{1}{2}} = \sqrt{D(\bar{x}_k, \lambda)} \hat{T}_k^- - \sqrt{B(\bar{x}_k, \lambda)}
\end{align*}
\]

VS: \[ \hat{a}_k(\hat{J}_-, \hat{J}_3)\Phi_0(\bar{x}_k) = 0 \rightarrow \Phi_{(0,0)}(\bar{x}_1, \bar{x}_2) \propto \sqrt{\frac{(\frac{-1}{2\hbar^2}) (\frac{1}{\lambda^2} + \frac{1}{2\hbar^2}) (\frac{1}{2\hbar^2})}{(\frac{1}{\lambda^2} + \frac{1}{2\hbar^2})!(\frac{1}{2\hbar^2})!}} \]

EF: \[ \Phi_{(n,m)}(\bar{x}_1, \bar{x}_2) \propto \left(\frac{-1}{\sqrt{2}\lambda}\right)^{n+m} C_n\left(\frac{\bar{x}_1}{\lambda\hbar} + \frac{1}{2\hbar^2}, \frac{1}{2\hbar^2}\right) C_m\left(\frac{\bar{x}_2}{\lambda\hbar} + \frac{1}{2\hbar^2}, \frac{1}{2\hbar^2}\right) \Phi_{(0,0)}(\bar{x}_1, \bar{x}_2) \]

Spectrum: \[ \hat{H}^{(2)}\Phi_{(n,m)}(\bar{x}_1, \bar{x}_2) = \hbar(n + m + 1)\Phi_{(n,m)}(\bar{x}_1, \bar{x}_2) \quad (n, m = 0, 1, \ldots) \]
What about the *conserved quantities*?
Discrete QM and the harmonic oscillator
Linking oQM and dQM: conserved quantities and superintegrability

What about the conserved quantities?

By applying the coproduct map $\Delta$ to the casimir:

$$\hat{C}^{(2)} = \Delta(\hat{C}) = \Delta \left( \frac{1}{2} (\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+) - \hat{J}_3^2 \right) = \hat{L}_z^2 - \hbar^2$$
Discrete QM and the harmonic oscillator

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The discrete analog of the angular momentum is obtained:

$$\hat{L}_z = \frac{i}{2\lambda^2} \left( \sqrt{2\lambda} \bar{x}_1 + 1 \hat{T}_1^- - \sqrt{2\lambda} \bar{x}_2 + 1 \hat{T}_2^- - \sqrt{2\lambda}(\lambda\hbar + \bar{x}_2) + 1 \left( \sqrt{2\lambda} \bar{x}_1 + 1 \hat{T}_{12}^{--} - \hat{T}_{2}^+ \right) \right.$$

$$\left. - \sqrt{2\lambda}(\bar{x}_1 + \lambda\hbar) + 1 \left( \hat{T}_1^+ - \sqrt{1 + 2\bar{x}_2\lambda\hat{T}_{12}^{+-}} \right) \right)$$

where

$$\hat{T}_{ij}^{\pm \mp} = \hat{T}_i^{\pm} \hat{T}_j^{\mp} = e^{\pm \lambda \hbar \partial_{\bar{x}_i} \mp \lambda \hbar \partial_{\bar{x}_j}} \quad (i, j = 1, 2)$$
Discrete QM and the harmonic oscillator

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- \sqrt{2\lambda}(\bar{x}_1 + \lambda \hbar) + 1 \left( \hat{T}_1^+ - \sqrt{1 + 2\lambda \bar{x}_2} \hat{T}_{12}^{+-} \right) \right)$$

where

$$\hat{T}_{ij}^{\pm \mp} \equiv \hat{T}_i^{\pm} \hat{T}_j^{\pm} = e^{\pm \lambda \hbar \partial_{\bar{x}_i} \mp \lambda \hbar \partial_{\bar{x}_j}} \quad (i, j = 1, 2)$$

$$[\hat{H}^{(2)}, \hat{L}_z] = 0 \Rightarrow \text{The discrete system is QMS!}$$
Discrete QM and the harmonic oscillator
Linking oQM and dQM: conserved quantities and superintegrability

What about the \textit{conserved quantities}?

By applying the coproduct map $\Delta$ to the casimir:

$$\hat{\mathcal{C}}^{(2)} \doteq \Delta(\hat{\mathcal{C}}) = \Delta \left( \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) - \hat{J}_3^2 \right) = \hat{L}_z^2 - \hbar^2$$

The discrete analog of the angular momentum is obtained:

$$\hat{L}_z = \frac{i}{2\lambda^2} \left( \sqrt{2\lambda\bar{x}_1 + 1} \hat{T}_1^- - \sqrt{2\lambda\bar{x}_2 + 1} \hat{T}_2^- - \sqrt{2\lambda(\lambda\hbar + \bar{x}_2) + 1} \left( \sqrt{2\lambda\bar{x}_1 + 1} \hat{T}_{12}^{-+} - \hat{T}_2^+ \right) 
- \sqrt{2\lambda(\bar{x}_1 + \lambda\hbar) + 1} \left( \hat{T}_1^+ - \sqrt{1 + 2\bar{x}_2\lambda} \hat{T}_{12}^{+-} \right) \right)$$

where

$$\hat{T}_{ij}^{\pm\mp} \doteq \hat{T}_i^{\pm} \hat{T}_j^{\mp} = e^{\pm\lambda\hbar \partial_{\bar{x}_i} + \lambda\hbar \partial_{\bar{x}_j}} \quad (i, j = 1, 2)$$

$$[\hat{H}^{(2)}, \hat{L}_z] = 0 \Rightarrow \text{The discrete system is QMS!}$$

In the continuous limit ($\lambda \to 0$) this difference expression collapses to the standard differential one (this is true for all the quantities we constructed)
Conclusions

To summarize the main points of this talk:

1. We have shown that it is possible to construct discrete quantum mechanical systems by means of canonical transformations establishing a link between oQM and dQM;

2. We constructed a discrete analog of the harmonic oscillator and solved the related spectral problem: the solution is given in terms of discrete orthogonal polynomials, the Charlier polynomials, that belong to the Askey scheme;

3. We extended the system to the 2D-case by using the coalgebra symmetry;

4. We constructed the discrete analog of the angular momentum ensuring the quasi-maximal superintegrability;

Besides this talk:

- The superintegrability has been obtained by constructing the discrete analog of the Fradkin Tensor;
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