Integrable dispersionless PDEs: IST and wave breaking

with S. V. Manakov: formal IST and all its consequences
with P. G. Grinevich and D. Wu: rigorous aspects of the IST
Also: Santucci and G. Yi (former students)

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Outline

Soliton PDEs vs Integrable dispersionless PDEs

Wave breaking in Nature

Some basic examples of dPDEs

IST for vector fields and dPDEs

Analytic description of wave breaking
INTEGRABLE SOLITON PDEs
Waves propagating in weakly nonlinear and dispersive media are well described by integrable soliton equations: KdV, NLS, ... 1) The Inverse Spectral Transform (IST) is the spectral method allowing one to solve the Cauchy problem for such PDEs, predicting that a localized disturbance evolves into a number of soliton pulses + radiation. Soliton = balance between nonlinearity and dispersion. 2) Soliton PDEs arise in hierarchies of commuting flows, sharing similar behavior. 3) Soliton PDEs are in low dimensions.

INTEGRABLE DISPERSIONLESS PDEs
1) Lax pair of integrable dPDEs is made of vector fields ⇒ can be in arbitrary dimensions; 2) Due to the lack of dispersion, dPDEs may exhibit wave breaking at finite time. 3) A novel IST for vector fields has been recently developed, to solve the Cauchy problem, allowing, in particular, to establish if, due to the lack of dispersion, the nonlinearity of the PDE is “strong enough” to cause the gradient catastrophe of localized multidimensional disturbances and to study the analytic details of such a wave breaking. 4) dPDs are intimately related to Twistor theory.
Wave breaking in Nature
Nonlinear **wave breaking** and **shock wave propagation** in multidimensions are very frequent natural phenomena:

1. Colour print by Hokusai, 1826; 2. Micro-explosion in air
3. Breaking waves in clouds; 4. Lagoon Nebula by Hubble

but difficult to be treated mathematically.
How does a water wave break?

The first breaking takes place in a point of spacetime, rapidly propagating in the transversal direction, and slowly propagating longitudinally. Can we describe analytically this phenomenon?
Equations describing such phenomena (equations of acoustics, hydrodynamics, plasma physics, etc..) are systems of PDEs, in general too complicated for extracting satisfactory informations on the phenomenon. Simplifying hypothesys, physically relevant, are useful, in the search for a simpler model equation, but able to capture the essence of the phenomenon:

**Universal properties of the model.** Consider a system of PDEs:

i) whose linear limit, at least in some approximation, is described by the wave equation. Then,

iii) studying the propagation of weakly nonlinear and quasi - one dimensional waves and

iv) neglecting dispersion and dissipation, one obtains the the \( dKP \) models with nonlinearity of degree \((m+1)\), in \( n + 1 \) dimensions, arising from many physical contexts, like acoustics, hydrodynamics, plasma physics, nonlinear optics, etc..:

\[
(u_t + u^m u_x)_x + \sum_{j=1}^{n-1} u_{y_jy_j} = 0, \quad dKP(m, n)
\]  

\( (1) \)
Commuting vector fields generate integrable PDEs in arbitrary dimensions [Zakharov Shabat ’79]

EXAMPLES

The commutation $[\hat{L}_1, \hat{L}_2] = 0$ of the vector fields:

$$\hat{L}_j = \partial_t^j + \lambda \partial_z^j + \vec{u}_z \cdot \nabla \vec{x}, \quad j = 1, 2$$  \hspace{1cm} (2)

is equivalent to the nonlinear vector PDE in $N + 4$ dimensions [Manakov-PMS 06]:

$$\vec{u}_{t_1 z_2} - \vec{u}_{t_2 z_1} + (\vec{u}_{z_1} \cdot \nabla \vec{x}) \vec{u}_{z_2} - (\vec{u}_{z_2} \cdot \nabla \vec{x}) \vec{u}_{z_1} = \vec{0},$$  \hspace{1cm} (3)

and its divergenceless reduction $\nabla \vec{x} \cdot \vec{u} = 0$.

1) Its deepest scalar Hamiltonian reduction in $2M + 4$ dimensions:

$$\theta_{t_2 z_1} - \theta_{t_1 z_2} + \{\theta_{z_1}, \theta_{z_2}\} = c(t_1, t_2, z_1, z_2)$$

$$\hat{L}_j = \partial_t^j + \lambda \partial_z^j + \{\theta_{z_j}, \cdot\}$$

$$\{f, g\} = \sum_{k=1}^{M} (f_{x_k} g_{x_{M+k}} - f_{x_{M+k}} g_{x_k})$$

i) the first heavenly equation

$N = 2; \partial_{t_1}, \partial_{t_2} = 0, \Rightarrow \{\theta_{x_1}, \theta_{x_2}\}_{z_1, z_2} = c(z_1, z_2)$

ii) the second heavenly equation (anti-self-duality + Einstein equations):

$N = 2; z_1 = x_1, z_2 = x_2 \Rightarrow \theta_{t_2 x_1} - \theta_{t_1 x_2} + \theta_{x_1 x_1} \theta_{x_2 x_2} - \theta_{x_1 x_2}^2 = c(t_1, t_2)$
Intimately related to SDYM via $U \rightarrow \vec{u} \cdot \nabla \vec{x}$

$L_i = \lambda \partial_{z_i} + \partial_{t_i} + U_{z_i}, \ i = 1, 2, \ [L_1, L_2] = 0 \ \Rightarrow$

$U_{t_1 z_2} - U_{t_2 z_1} + [U_{z_1}, U_{z_2}] = 0$

Recursion operator [Marvan, Sergyeyev 2012] and bi-Hamiltonian structures follow directly from those of the SDYM [Bruschi, Levi, Ragnisco 1981]:

$SDYM : \ \Theta_1 \equiv \partial_{z_1}, \ \Theta_2 \equiv \partial_{t_1} + ad(U_{z_1}), \ \Phi \equiv \Theta_2(\Theta_1)^{-1}$

$\Rightarrow \ R = \Theta_2(\Theta_1)^{-1}, \ \Theta_2 \vec{f} = \vec{f}_{t_1} + (\vec{u}_{z_1} \cdot \nabla \vec{x}) \vec{f} - (\vec{f} \cdot \nabla \vec{x}) \vec{u}_{z_1}$
2) The system of two nonlinear PDEs in $2+1$ dimensions [Manakov and PMS ’06]:

$$u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} = 0,$$

$$v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} = 0,$$

(4)

commutation condition $[\tilde{L}_1, \tilde{L}_2] = 0 \ \forall \lambda$, involving also $\partial_\lambda$:

$$\tilde{L}_1 \equiv \partial_y + (\lambda + v_x) \partial_x - u_x \partial_\lambda,$$

$$\tilde{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y) \partial_x + (-\lambda u_x + u_y) \partial_\lambda,$$

(5)

it describes the most general integrable Einstein-Weyl metric structure [Dunajski ’08; Dunajski, Ferapontov, Kruglikov ’14]

1a) The $v = 0$ reduction of (4), the celebrated dKP equation

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R},$$

(6)

describing the evolution of small amplitude, nearly one-dimensional waves when dispersion and dissipation are negligible. It is a basic prototype model in the description of multidimensional wave breaking [Manakov, PMS 2008].
Commutation condition for a pair of Hamiltonian 2D vector fields:

\[
\hat{L}_1 \equiv \partial_y + \lambda \partial_x - u_x \partial_\lambda = \partial_y + \{H_1, \cdot\}(\lambda, x),
\hat{L}_2 \equiv \partial_t + (\lambda^2 + u) \partial_x + (-\lambda u_x + u_y) \partial_\lambda = \partial_t + \{H_2, \cdot\}(\lambda, x),
\]

\[H_1 = \frac{\lambda^2}{2} + u(x, y), \quad H_2 = \frac{\lambda^3}{3} + \lambda u - \partial_x^{-1} u_y,\]  

(7)

dKP hierarchy:

\[H_{nt_m} - H_{mt_n} + \{H_m, H_n\}(\lambda, x) = 0, \quad H_n \equiv \frac{1}{n} (f^n)_{\geq 0},\]  

(8)

where \(f\) is the eigenfunction analytic in a neighbourough of \(\lambda = \infty\), with the expansion:

\[f = \lambda + u \lambda^{-1} - \partial_x^{-1} (u_y) \lambda^{-2} + \sum_{j \geq 3, j \in \mathbb{Z}} q_j \lambda^{-j},\]  

(9)

1b. The \(u = 0\) reduction of (4), the Pavlov equation

\[v_{xt} + v_{yy} = v_y v_{xx} - v_x v_{xy}, \quad v = v(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R},\]  

(10)

associated with the non-Hamiltonian one-dimensional vector fields

\[
\hat{L}_1 \equiv \partial_y + (\lambda + v_x) \partial_x,
\hat{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x - v_y) \partial_x.
\]  

(11)
IST for VECTOR FIELDS [Manakov and PMS ’05-06]

Basic example: the dKP system

\[
\begin{align*}
    u_{xt} + u_{yy} &= -(uu_x)_x - v_x u_{xy} + v_y u_{xx}, \quad u, v \in \mathbb{R}, \ x, y, t \in \mathbb{R}, \\
    v_{xt} + v_{yy} &= -u v_{xx} - v_x v_{xy} + v_y v_{xx}
\end{align*}
\]  

(12)

describing the most general Einstein-Weyl metric, and its Lax pair formulation

\[
\begin{align*}
    \hat{L}_1 \psi &= 0, \quad \hat{L}_2 \psi = 0, \quad \Leftrightarrow \quad [\hat{L}_1, \hat{L}_2] = 0
\end{align*}
\]

\[
\begin{align*}
    \tilde{L}_1 &\equiv \partial_y + (\lambda + v_x)\partial_x - u_x\partial_\lambda, \\
    \tilde{L}_2 &\equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda
\end{align*}
\]  

(13)

Novel features of the IST for vector fields

Since the Lax pair is made of vector fields (Hamiltonian in the dKP (\(v = 0\) reductions):

1) The space of eigenfunctions is a ring: if \(f_1, f_2\) are two eigenf.s, then an arbitrary differentiable function \(F(f_1, f_2)\) of them is also an eigenf.

2) In the Hamiltonian reductions, the space of eigenfunctions is also a Lie algebra, whose Lie bracket is the natural Poisson bracket: if \(f_1, f_2\) are two eigenf.s, then their Poisson bracket \(\{f_1, f_2\}\) is also an eigenf.
Cauchy problem for rapidly decreasing real 2D waves
\( \hat{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x - u_x\partial_\lambda \)

If \( f \) is a solution of \( \hat{L}_1 f = 0 \), then
\[
f(x, y, \lambda) \to f_\pm(\xi, \lambda), \quad y \to \pm\infty,
\]
\[
\xi := x - \lambda y;
\]  
(14)
i.e., asymptotically, \( f \) is an arbitrary function of \( \xi = (x - \lambda y) \), and \( \lambda \).

Real (Jost) eigenfunctions \( \vec{\phi}_\pm(x, y, \lambda) \):
\[
\vec{\phi}_\pm \equiv \begin{pmatrix} \phi_{1\pm}(x, y, \lambda) \\ \phi_{2\pm}(x, y, \lambda) \end{pmatrix} \to \begin{pmatrix} \lambda \\ -\lambda y + x \end{pmatrix} \equiv \vec{\xi}, \quad y \to \pm\infty.
\]  
(15)
intimately related to the system of real ODEs
\[
\frac{dx}{dy} = \lambda + v_x(x, y), \quad \frac{d\lambda}{dy} = -u_x(x, y)
\]  
(16)
defining the characteristics of \( \hat{L}_1 \).
If the potentials \((u, v)\) are sufficiently regular, the solution \((x(y), \lambda(y))\) of the ODE (16) exists unique globally in the (time) variable \(y\), with the following free particle asymptotic behavior

\[
x(y) \to \lambda_\pm y + x_\pm, \quad \lambda(y) \to \lambda_\pm, \quad y \to \pm \infty,
\]

reducing to the asymptotics

\[
x(y) \to \lambda y + x_\pm, \quad \lambda(y) = \lambda = \text{constant}, \quad y \to \pm \infty,
\]

in the Pavlov reduction \(u = 0\). Once the asymptotics \(\lambda_\pm, x_\pm\) are constructed in terms of the initial data \(x_0 = x(y_0), \lambda_0 = \lambda(y_0)\) of the ODE: \(\lambda_\pm(x_0, y_0, \lambda_0), x_\pm(x_0, y_0, \lambda_0)\), the real eigenfunctions \(\vec{\phi}_\pm\), that are particular constants of motion of motion of the ODE, are given by

\[
\vec{\phi}_\pm(x_0, y_0, \lambda_0) = (x_\pm(x_0, y_0, \lambda_0), \lambda_\pm(x_0, y_0, \lambda_0)).
\]
Another important ingredient of the formalism is given by the complex eigenfunction $\vec{\psi}$, defined by the asymptotics

$$\vec{\psi}(y, \vec{x}, \lambda) \sim \vec{\xi}, \; x^2 + y^2 \to \infty, \; \lambda \notin \mathbb{R},$$  \hspace{1cm} (20)

analytic for $\lambda \notin \mathbb{R}$, having continuous boundary values $\vec{\psi}^\pm(x, y, \lambda), \; \lambda \in \mathbb{R}$ from above and below the real $\lambda$ axis, with the following asymptotics for large complex $\lambda$:

$$\vec{\psi}^\pm(x, y, \lambda) = \vec{\xi} + \frac{1}{\lambda} \vec{U}(x, y) + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \; |\lambda| >> 1,$$

$$\vec{U}(x, y) \equiv \begin{pmatrix} -yu(x, y) - v(x, y) \\ u(x, y) \end{pmatrix}. \hspace{1cm} (21)$$
Scattering and spectral data. The \( y = +\infty \) limit of \( \vec{\phi}_- \) defines the natural (\( y \) - time) scattering vector \( \vec{\sigma} \) for \( \hat{L}_1 \):

\[
\lim_{y \to +\infty} \vec{\phi}_-(x, y, \lambda) \equiv \vec{S}(\vec{\xi}) = \vec{\xi} + \vec{\sigma}(\vec{\xi}). \tag{22}
\]

Since the space of eigenfunctions is a ring, the eigenfunctions \( \vec{\psi}_\pm \) for \( \lambda \in \mathbb{R} \) can be expressed in terms of the real eigenfunctions \( \vec{\phi}_\pm \), and this expression defines the spectral data \( \vec{\chi}_\beta^\pm(\vec{\xi}, \lambda) \):

\[
\vec{\psi}_\pm(x, y, \lambda) = \vec{\mathcal{K}}^\pm_{\beta}(\vec{\phi}_-(x, y, \lambda)) = \vec{\mathcal{K}}^\pm_{\beta}(\vec{\phi}_+(x, y, \lambda)), \quad \lambda \in \mathbb{R},
\]

\[
\vec{\mathcal{K}}^\pm_{\beta}(\vec{\xi}) \equiv \vec{\xi} + \vec{\chi}_\beta^\pm(\vec{\xi}), \quad \vec{\xi} = (\xi, \lambda), \tag{23}
\]

where \( \vec{\chi}_\beta^+(\vec{\xi}) \) and \( \vec{\chi}_\beta^-(\vec{\xi}) \) are analytic wrt the first argument \( \xi \) respectively in the upper and lower halves of the complex \( \xi \) - plane, as a consequence of the analyticity properties of \( \vec{\psi}_\pm \).
Evaluating
\[
\vec{K}^\pm(\vec{\phi}^-(x, y, \lambda)) = \vec{K}^-_+(\vec{\phi}^+(x, y, \lambda)), \quad \lambda \in \mathbb{R}, \\
\vec{K}^{\pm}_\beta(\vec{\xi}) \equiv \vec{\xi} + \vec{\chi}^{\pm}_\beta(\vec{\xi}), \quad \vec{\xi} = (\xi, \lambda),
\] (24)

at \( y = +\infty \), one obtains the following linear Riemann - Hilbert (RH) problem with a shift:

\[
\vec{\sigma}(\xi, \lambda) + \vec{\chi}^+_-((\xi + \vec{\sigma}(\xi, \lambda)) - \vec{\chi}^-_+(\xi, \lambda) = \vec{0}, \\
|\vec{\chi}^{\pm}_\beta(\xi, \lambda)| = \mathcal{O}(\xi^{-1}), \quad \xi \sim \infty
\] (25)
equivalent to a linear Fredholm integral equation, allowing one to uniquely construct the spectral data \( \vec{\chi}^+_- \) and \( \vec{\chi}^-_+ \) from the scattering data \( \vec{\sigma} \), under the hypothesis that the mapping \( \xi \rightarrow \xi + \sigma_1(\xi, \lambda) \) be invertible.

Reality conditions (from reality of the potentials):

\[
(u, v) \in \mathbb{R}^2 \Rightarrow \vec{\phi}^\pm \in \mathbb{R}^2, \quad \vec{\psi}^- = \overline{\vec{\psi}^+}, \quad \vec{\sigma} \in \mathbb{R}^2, \quad \vec{\chi}^-_\alpha = \overline{\vec{\chi}^+_\alpha}, \quad \lambda \in \mathbb{R}. \] (26)
Two inverse problems

The first inversion (the reconstruction of $\vec{\phi}_-$ from the spectral data $\vec{\chi}_-^\pm$) is provided by the nonlinear integral equation

$$\vec{\phi}_-(x, y, \lambda) + H_\lambda \vec{\chi}_-^I(\vec{\phi}_-(x, y, \lambda)) + \vec{\chi}_-^R(\vec{\phi}_-(x, y, \lambda)) = \vec{\xi}, \tag{27}$$

where $\vec{\chi}_-^R$ and $\vec{\chi}_-^I$ are the real and imaginary parts of $\vec{\chi}_-^\pm$, and $H_\lambda$ is the Hilbert transform operator wrt $\lambda$

$$H_\lambda f(\lambda) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(\lambda')}{\lambda - \lambda'} d\lambda'. \tag{28}$$

Equation (27) expresses the fact that the RHS of (23) for $\vec{\psi}^+$ is the boundary value of a function analytic in the upper half $\lambda$ plane.

Once $\vec{\phi}_-$ is reconstructed from $\vec{\chi}_-^\pm$ solving the nonlinear integral equation (27), equations (23) give $\vec{\psi}^\pm$, and $(u, v)$ is finally reconstructed from

$$u(x, y) = \lim_{\lambda \to \infty} \left( \lambda(\psi_2^-(x, y, \lambda) - \lambda) \right),$$

$$v(x, y) = -yu - \lim_{\lambda \to \infty} \left( \lambda(\psi_1^-(x, y, \lambda) - (x - \lambda y)) \right), \tag{29}$$
A second inverse problem can be obtained eliminating the real eigenfunctions from the first of equations (23) for $\vec{\psi}^\pm$, obtaining a 2 vector nonlinear RH (NRH) problem on the real line:

\begin{align*}
\psi_1^+(\lambda) &= R_1 (\psi_1^-(\lambda), \psi_2^-(\lambda)), \quad \lambda \in \mathbb{R}, \\
\psi_2^+(\lambda) &= R_2 (\psi_1^-(\lambda), \psi_2^-(\lambda)), \\
\psi_1^-(\lambda) &= -y\lambda + x + O(\lambda^{-1}), \quad \psi_2^+(\lambda) = \lambda + O(\lambda^{-1}), \quad \lambda \sim \infty.
\end{align*}

(30)

for the RH data $\vec{R}$, constructed, via algebraic manipulation, from the spectral data. Once the analytic eigenfunctions are reconstructed through the solution of the NRH problem (30), the solution of the nonlinear PDE (4) is obtained from (29). We remark that, in the two basic reductions, the RH data are constrained as follows:

\begin{align*}
R_2(\zeta_1, \zeta_2) &= \zeta_2, \quad \text{Pavlov reduction}, \\
\{R_1, R_2\}_{\zeta_1, \zeta_2} &= 1, \quad \text{dKP reduction}.
\end{align*}

(31)
Evolution of the spectral data. The evolution of the scattering, spectral, and RH data is described by the following simple formula [Manakov, PMS 2006,07]:

\[ \Sigma_1(\xi, \lambda, t) = \Sigma_1(\xi - \lambda^2 t, \lambda, 0) \] (32)

for the Pavlov equation, and

\[ \begin{align*}
\Sigma_1(\xi, \lambda, t) &= t(\Sigma_2(\xi - \lambda^2 t, \lambda, 0))^2 + \Sigma_1(\xi - \lambda^2 t, \lambda, 0), \\
\Sigma_2(\xi, \lambda, t) &= \Sigma_2(\xi - \lambda^2 t, \lambda, 0)
\end{align*} \] (33)

for the dKP equation.

We remark that, from the eigenfunctions \( \vec{\phi}_\pm, \vec{\psi}_\pm \) of \( \hat{L}_1 \), one can constructs the common eigenfunctions \( \Phi_\pm, \Psi_\pm \) of \( \hat{L}_1 \) and \( \hat{L}_2 \) through the formulae

\[ \begin{align*}
\Phi_{\pm 1} &= \phi_{\pm 1} - t(\phi_{\pm 2})^2, & \Phi_{\pm 2} &= \phi_{\pm 2}, \\
\Psi_{\pm 1} &= \psi_{\pm 1} - t(\psi_{\pm 2})^2, & \Psi_{\pm 2} &= \psi_{\pm 2}
\end{align*} \] (34)
Nonlinear Riemann - Hilbert dressing. Let $\vec{\Psi}^\pm(\lambda)$ be the solutions of the following 2 vector NRH problem on the line

$$\vec{\Psi}^+(\lambda) = \vec{R}\left(\vec{\Psi}^-(\lambda)\right), \; \lambda \in \mathbb{R}, \quad (35)$$

with the normalization

$$\vec{\Psi}^\pm(\lambda) = \left(-t\lambda^2 - y\lambda + x - 2ut\right) + \vec{O}(\lambda^{-1}), \; \lambda \sim \infty, \quad (36)$$

for the RH data $\vec{R}(\vec{\zeta}) = (R_1(\vec{\zeta}), R_2(\vec{\zeta}))$, $\vec{\zeta} \in \mathbb{C}^2$. Then $\vec{\Psi}^\pm(\lambda)$ are eigenfunctions of $\hat{L}_j \; j = 1, 2$: $\hat{L}_j \vec{\Psi}^\pm = \vec{0}$, $j = 1, 2$, and one obtains the following spectral characterization of the solution $u$:

$$u = F(x - 2ut, y, t) \in \mathbb{R}, \quad (37)$$

where the spectral function $F$, defined by

$$F(\xi, y, t) = -\int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R_2\left(\psi_1^-(\lambda; \xi, y, t), \psi_2^-(\lambda; \xi, y, t)\right), \quad (38)$$

is connected to the initial data via the direct problem [?].
The longtime behavior of dKP solutions
Let $t >> 1$ and
\[
x = \xi + v_1 t, \quad y = v_2 t, \\
\xi - 2ut, \ v_1, \ v_2 = O(1), \ v_2 \neq 0, \ t >> 1.
\]
(39)

On the parabola
\[
x + \frac{y^2}{4t} = \xi \ (v_1 = -\frac{v_2^2}{4}),
\]
(40)
the longtime behaviour of the solutions of the dKP equation is given by
\[
u = \frac{1}{\sqrt{t}} G\left(x + \frac{v_1}{4t}, \frac{v_2}{2t}\right) (1 + o(1)),
\]
(41)
where $a_j(\mu : \xi, \eta)$ solve “asymptotic” RH problem on the $\mu$ real axes:
\[
\vec{A}^{\pm}(\mu; \xi, \eta) = \vec{A}^{-}(\mu; \xi, \eta) + \vec{R}(\vec{A}^{-}(\mu; \xi, \eta)), \ \mu \in \mathbb{R},
\]
\[
\vec{a}^{\pm}(\mu; \xi, \eta) = \left( \begin{array}{c}
\xi + \mu^2 \\
\eta
\end{array} \right) + \vec{a}(\mu; \xi, \eta).
\]
(42)

Small initial data start evolving according to $u_{tx} + u_{yy} = 0$. Only in the longtime regime the nonlinear term becomes relevant, causing the breaking of the small localized initial wave in a point of the parabola.
NO breaking mechanism instead for the the Pavlov equation [Manakov and PMS 09]

**RH INVERSE PROBLEM (Pavlov):** \( \mathcal{R}(\zeta_1, \zeta_2) \Rightarrow v(x, y, t) \)

In the Pavlov reduction:

\[
\mathcal{R}(\mathcal{R}(\zeta)) = \zeta, \quad \forall \zeta \in \mathbb{C}^2,
\]

\[
\mathcal{R}_1(\zeta) = \zeta_1 \Rightarrow \psi_1^+(\lambda) = \psi_1^-(\lambda) = \lambda \Rightarrow u = 0
\]  

(43)

the nonlinear RH problem becomes scalar for \( \psi_2^\pm(\lambda) \):

\[
\Phi^+ (\lambda) = \mathcal{R}(\lambda, \psi^- (\lambda)), \quad \lambda \in \mathbb{R}
\]

(44)

with normalization:

\[
\Phi^\pm (\lambda) = -\lambda^2 t - \lambda y + x + O\left(\frac{1}{\lambda}\right),
\]

(45)

Then

\[
v(x, y, t) = \int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R\left(\lambda, \Phi^-(\lambda; x, y, t)\right).
\]

(46)
The IST allows one to show that solutions $u(x, y, t)$ of dKP depend on $x$ through the combination $x - 2ut$; i.e., these solutions can be written in the characteristic form

$$u = F(\zeta, y, t), \quad \zeta = x - 2F(\zeta, y, t)t,$$

$$F(x, y, 0) = u(x, y, 0),$$

(47)

in analogy with the case of the Riemann equation $u_t + u^m u_x = 0$, for which the dependence of the solution $u(x, t)$ on $x$ is through the combination $x - u^m t$. For this reason, the IST for dKP can be viewed as a generalization of the method of characteristics. The formulation (47) becomes explicit in the small field limit

$$u \sim \tilde{u}(x - 2ut, y, t),$$

$$\tilde{u}_{xt} + \tilde{u}_{yy} = 0, \quad \tilde{u}(x, y, 0) = u(x, y, 0).$$

(48)

The formulation (47) has allowed one to study in an analytically explicit way the interesting features of the gradient catastrophe of two dimensional waves at finite time and in the longtime regime in terms of the initial data [Manakov, PMS 2008, 2011, 2012].
Given $F$ from the inverse problem, we solve (47b) wrt $\zeta$: $\zeta(x, y, t)$ and we replace it in (47a), obtaining the solution $u = F(\zeta(x, y, t), y, t)$ of the Cauchy problem for dKP. Therefore the Singularity Manifold (SM) of dKP is the two-dimensional manifold characterized by the equation

$$S(\zeta, y, t) \equiv 1 + 2F_\zeta(\zeta, y, t)t = 0; \quad (49)$$

Since

$$\nabla(x,y)u = \frac{\nabla(\zeta,y)F(\zeta,y,t)}{1 + 2F_\zeta(\zeta,y,t)t}, \quad (50)$$

the gradient of the wave becomes infinity on the SM, and the wave “breaks”.

The first time $t_b$ at which $S = 0$ in a point $\bar{\zeta}_b = (\zeta_b, y_b)$ of the $(\zeta, y)$-plane:

$$t_b \equiv \text{global min } \ddot{t}(\zeta, y) = \ddot{t}(\zeta_b, y_b), \quad \Rightarrow \quad 1 + 2F_\zeta(\bar{\zeta}_b, t_b)t_b = 0; \quad (51)$$
conditions characterizing the breaking point \((\vec{\zeta}_b, t_b)\):

\[1 + 2t_b F_\zeta(\vec{\zeta}_b, t_b) = 0\]
\[F_\zeta(\vec{\zeta}_b, t_b) < 0, \quad F_\zeta(\vec{\zeta}_b, t_b) + t_b F_{\zeta t}(\vec{\zeta}_b, t_b) < 0,\]
\[F_{\zeta\zeta}(\vec{\zeta}_b, t_b) = F_{\zeta y}(\vec{\zeta}_b, t_b) = 0,\]
\[F_{\zeta\zeta\zeta}(\vec{\zeta}_b, t_b) > 0, \quad \beta \equiv F_{\zeta\zeta\zeta}(\vec{\zeta}_b, t_b) F_{\zeta y y}(\vec{\zeta}_b, t_b) - F_{\zeta y}(\vec{\zeta}_b, t_b)^2 > 0.\]  

(52)

At \(t = t_b\) the wave breaks in the point \(\vec{x}_b = (x_b, y_b)\) of the \((x, y)\) - plane defined by

\[x_b = \zeta_b + 2F(\vec{\zeta}_b, t_b)t_b.\]  

(53)

Therefore, generically, the solution breaks at the finite point \((x_b, y_b, t_b)\) of space-time; in addition, due to (50), all derivatives of \(u\) blow up at \((x_b, y_b, t_b)\), except the derivative along the “transversal line of breaking”, characterized by the vector field \(\hat{V} = 2F_y t \partial_x + \partial_y\), for which

\[\hat{V}u = F_y.\]

(54)
Now we study the analytic behavior of the dKP solution near breaking, evaluating the characteristic equations $\zeta = x - 2F(\zeta, y, t) t$ in the regime:

$$
x = x_b + x', \quad y = y_b + y', \quad t = t_b + t', \quad \zeta = \zeta_b + \zeta',
$$

(55)

where $x', y', t', \zeta'$ are small, obtaining, at the leading order, the cubic

$$
\text{Cubic}(\zeta'; x', y', t') \equiv \zeta'^3 + a(y')\zeta'^2 + b(y', t')\zeta' - \gamma X(x', y', t') = 0,
$$

(56)

where

$$
a(y') = \frac{3F_{\zeta\zeta\zeta\zeta}}{F_{\zeta\zeta\zeta}} y', \quad b(y', t') = \gamma [2(F_\zeta + t_b F_{\zeta t}) t' + F_{\zeta y y} t_b y'^2],
$$

$$
X(x', y', t') = x' - 2F(\zeta_b, y, t) t' - 2 [F(\zeta_b, y, t) - F] t_b \sim
$$

$$
x' + \frac{F_y}{F_\zeta} y' - 2(F + t_b F_t) t' - \frac{F_{yy}}{2|F_\zeta|} y'^2 - 2(F_y + t_b F_{yt}) y' t' - \frac{F_{yyy}}{6|F_\zeta|} y'^3,
$$

$$
\gamma = \frac{6|F_\zeta|}{F_{\zeta\zeta\zeta}}.
$$

(57)

corresponding to the maximal balance

$$
|\zeta'|, |y'| = O(|t'|^{1/2}), \quad X = O(|t'|^{3/2}).
$$

(58)
The Function $S$ reads, at the leading order,

$$S = 2(F_\zeta + t_b F_{\zeta t}) t' + \left( F_{\zeta \zeta \zeta} \zeta'^2 + 2F_{\zeta \zeta y} y' \zeta' + F_{\zeta y y} y'^2 \right) t_b. \tag{59}$$

The solution is therefore described, around the breaking point, by

$$u = F(\zeta_b + \zeta', y_b + y', t_b + t') \sim F + F_\zeta \zeta' + F_y y' + O(t'), \tag{60}$$

$$\text{Cubic} (\zeta'; x', y', t') = 0.$$ [Manakov, PMS 2008, 2011, 2012]

Equivalently, as in $1+1$, one eliminates $\zeta'$, obtaining a cubic for $u$:

$$\text{Cubic} \left( \frac{u - F - F_y y'}{F_\zeta}; x', y', t' \right) = 0. \tag{61}$$
After breaking. If \( t > t_b \ (t' > 0) \), the SM in space-time coordinates is given by the zero discriminant condition \( \Delta(x', y', t') = 0 \), delimiting a compact region of the \((x,y)\) plane with two cusp points \((x^+_C(t'), y^+_C(t'))\):

\[
y^+_C(t') \equiv \pm |F_\zeta| \sqrt{2F_\zeta \zeta \zeta \beta \sqrt{t'}} ,
\]

\[
x^+_C(t') \equiv -\frac{F_y}{F_\zeta} y^+_C(t') + \left(F + \left(\frac{F_y}{F_\zeta}\right)^2\right) t' + \frac{F_{yy}}{2|F_\zeta|} y^+_C(t')^2 + 2(F_y + t_b F_{yt}) y^+_C(t') t' + \frac{F_{yyy}}{6|F_\zeta|} y^+_C(t')^3 \pm \frac{2}{\gamma} \left(-\frac{a(y')b(y', t')}{18} - \frac{a(y')}{3} Q(y', t') \pm |Q(y', t')|^{3/2}\right).
\]
We end this section remarking that, since

\[ x_B^+(y', t') - x_B^-(y', t') = O(t'^{3/2}), \tag{66} \]

it follows that the transversal width of the three valued region, estimated by the distance of the two cusps, is \( O(t'^{1/2}) \), while the longitudinal width is \( O(t'^{3/2}) \). Consequently, the three valued region develops, from the breaking point \((x_b, y_b)\), with infinite speed in the transversal direction, and with zero speed in the longitudinal one. Typical shapes of the multivalued region for \( y \)-symmetric data:
References.

